Right-angled Artin groups and their right-angled Artin subgroups

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Γ a graph.

- finite
- undirected
- no double edges
- no edges of the form (v,v).

$V = V(\Gamma)$ vertices

$E \subset V \times V$ edges

$A(\Gamma)$ is the right-angled Artin group on Γ.

$$A(\Gamma) = \langle V \mid [v_i, v_j] = 1 \Leftrightarrow (v_i, v_j) \in E \rangle.$$

$(V = S.\ (v_i, v_j) \in E \Rightarrow m_{ij} = 2.\ (v_i, v_j) \notin E \Rightarrow m_{ij} = \infty$

Examples: $\Gamma = K_n$, $A(\Gamma) \cong \mathbb{Z}^n$

$\Gamma = D_n$, $A(\Gamma) \cong F_n$

$\Gamma =$  $A(\Gamma) \cong F_2 \times F_2$

Right-angled Artin groups interpolate between free & abelian groups.

What are their subgroups?

① If $H < F_n$, then H is free (Nielsen-Schreir).

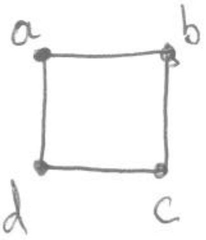
② If $H < \mathbb{Z}^n$ then H is abelian.

Note: in case ①, $\operatorname{rk} H$ can be arbitrarily large.
in case ②, $\operatorname{rk} H \leq n$.

Is every f.g. subgroup of a RAAG also a RAAG?

NO!

Example: (M. Casals)



$$F_2 \times F_2 \cong \langle a, c\rangle \times \langle b, d\rangle$$

$$\vee$$

$$H = \langle \underset{\overset{\parallel}{x}}{ab},\ c, d\rangle$$

Observe: $[x, c],\ [x, d] \neq 1$

$[c, d] = 1$

$[c^x, d] = [c, d] = 1$, but $(c, d) \underset{\text{conj}}{\not\sim} (c^x, d)$.

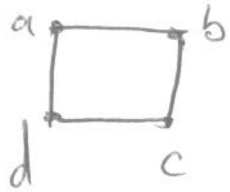
Is H (abstractly) isomorphic to a RAAG?

H is at most 3-generated, so if $H \cong A(\Gamma)$ then $|V(\Gamma)| \leq 3$.

Possibilities:

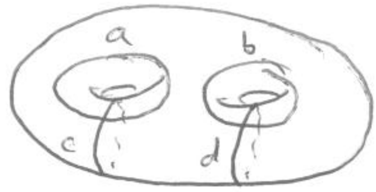
Γ	$A(\Gamma)$
	$\mathbb{Z}$
	F_2
	F_3
	$\mathbb{Z}^2$
	$F_2 \times \mathbb{Z}$
	$\mathbb{Z}^3$
	$\mathbb{Z}^2 * \mathbb{Z}$

$H = \langle x = ab, c, d \rangle$, cont'd.



$[c,d] = 1 \implies H$ is not free

$[x,c] \neq 1 \implies H$ is not abelian



2 copies of $\mathbb{Z}^2$: $\langle c, d \rangle$, $\langle c^x, d \rangle$ are not conjugate commensurable.

$\implies H \not\cong \mathbb{Z}^2 * \mathbb{Z}$.

Finally, H has no center, so $H \not\cong F_2 \times \mathbb{Z}$.

What if we insist that H be: normal? f.p.? FP_{∞}?

Still, H may not be a RAAG.

$\Gamma \rightsquigarrow F(\Gamma)$ = flag complex.

$A(\Gamma) \longrightarrow \mathbb{Z}$, $v_i \longmapsto 1$ Kernel $= B_\Gamma$

Thm (Bestvina–Brady): B_Γ admits a $K(G,1)$ w/ finite k-skeleton $\iff F(\Gamma)$ is k-connected.

What are some RAA-subgroups of RAAGs?

If $\Lambda < \Gamma$ then $A(\Lambda) < A(\Gamma)$.

Something more sophisticated?

Let $v \in \Gamma$.

Thm: $\langle \{V(\Gamma)\} \setminus v, \{V(\Gamma)\}^{v} \setminus v, v^2 \rangle \cong A(\Gamma \cup_{st(v)} \Gamma)$.

Construction: Extension graph Γ^e.

Def. 1: $V(\Gamma) \subset A(\Gamma) \circlearrowleft A(\Gamma)$.

$V(\Gamma^e)$ = pts. in $A(\Gamma)$-orbits of $V(\Gamma)$.

$E(\Gamma^e)$ = pts. in $A(\Gamma)$-orbits of $E(\Gamma)$.

Note: $E(\Gamma) \subset V(\Gamma) \times V(\Gamma) \subset A(\Gamma) \times A(\Gamma) \circlearrowleft A(\Gamma)$ diagonal action.

Def. 2: $\Gamma \bigcup_{st(v)} \Gamma$. Repeat, varying v.

These constructions are equivalent.

Thm: If $\Lambda < \Gamma^e$ finite, then $A(\Lambda) < A(\Gamma)$.

"More robust" proof.

Thm (K): Let $\varphi_1, \ldots, \varphi_k$ be Dehn twists about distinct isotopy classes of essential s.c.c.s.

$$\Gamma = \Gamma(\varphi_1, \ldots, \varphi_k) := \begin{cases} V(\Gamma) = \{\varphi_i\} \\ E(\Gamma) = \begin{cases} \text{fix hyp. metric, represent curves by geodesics, } (\varphi_i, \varphi_j) \in E \\ \Longleftrightarrow \operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j) \end{cases} \end{cases}$$

$\exists N$ s.t. $\forall n \geq N$, $\langle \varphi_1^n, \ldots, \varphi_k^n \rangle \cong A(\Gamma)$.

Equivalent tool: M. Clay, C. Leininger, J. Mangahas.

pf of Thm: Choose curves w/ coincidence graph $= \Gamma$. Note: $T_\alpha^{T_\beta} = T_{T_\beta(\alpha)}$.

If $\{\alpha_1, \dots, \alpha_k\}$ are the curves and α_i is fixed, then $\exists N$ s.t. $\forall n \geq N$, $\langle \{T_{\alpha_j}^n\} \rangle \cong A(\Gamma)$

and $\Gamma(\{\alpha_1, \dots, \alpha_k, T_{\alpha_i}^n(\alpha_1), \dots, T_{\alpha_i}^n(\alpha_k)\}) = \Gamma \cup_{St(\alpha_i)} \Gamma$.

Apply Thm (K). $\square$

What new information do we get?

 P_4

Cor.

If F is any finite forest then $A(F) < A(P_4)$.

Why? P_4^e is a regular ∞-valence tree, together w/ ∞-many degree 1 vertices attached to each vertex.

What does Γ^e look like?

P_4 ✓

 $\rightsquigarrow K_{\infty,\infty}$.

Some systematic descriptions of the geometry of Γ^e:

1. Γ connected $\Rightarrow \Gamma^e$ connected.
2. $\Gamma = \Gamma_1 * \Gamma_2 \Rightarrow \Gamma^e = \Gamma_1^e * \Gamma_2^e$
3. $\Gamma = \underset{\neq\emptyset}{\Gamma_1} \sqcup \underset{\neq\emptyset}{\Gamma_2} \Rightarrow \Gamma^e = \left(\coprod_\infty \Gamma_1^e\right) \sqcup \left(\coprod_\infty \Gamma_2^e\right)$
4. $\Lambda < \Gamma \Rightarrow \Lambda^e < \Gamma^e$.
5. Γ^e is finite $\Leftrightarrow \Gamma$ is complete.
6. Γ^e has finite diameter $\Leftrightarrow \Gamma$ splits as a nontrivial jo
7. Suppose $A(\Gamma)$ has no center. Then $\Gamma^e \setminus St(v)$ is disconnected $\forall v \in \Gamma^e$
8. Γ^e is a quasi-forest for any graph Γ.

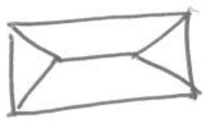
Let C_n = cycle of length n.

9 If $C_n, \overline{C_n} \not\leq \Gamma$ for each $n \geq 5$ (weakly chordal), then $C_n \not\leq \Gamma^e$.

Consequence: It is easy to predict the occurrences of long cycles in Γ^e.

Why should we care?

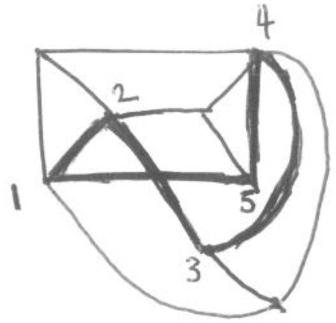
$$\overline{C_5} \cong C_5$$

$\overline{C_6} \cong$ 

Thm (Kim): $A(C_5) < A(\overline{C_6})$.

Old question (Gordon-Reid): If $A(\Gamma)$ contains A(long cycle), does Γ contain a long cycle? (No)

Proof of Kim's result, using new machinery:



$\square$

How about a converse?

Thm: Suppose Γ is Δ-free and that $A(\Lambda) < A(\Gamma)$. Then $\Lambda < \Gamma^e$.

Some consequences:

If Γ is weakly chordal and Δ-free then $A(C_n) \not< A(\Gamma)$ for $n \geq 5$.

If $A(\Lambda) < F_2 \times F_2$ then $A(\Lambda) \cong \begin{cases} F_k \times F_l \\ \text{or} \\ F_n \end{cases}$.

Suppose $A(C_m) < A(C_n)$, $n \geq 4$.

Then $m = n + k(n-4)$, $k \geq 0$.

Sub consequences: $A(C_m) < A(C_5)$ $\forall m \geq 5$.

$A(C_m) < A(C_6) \Leftrightarrow m \geq 6$ and even.

$A(C_4) < A(C_n) \Leftrightarrow n = 4$.

Special case of Kambites' theorem: $A(C_4) < A(\Gamma) \Leftrightarrow$ Γ contains C_4.

A slightly more general version of the previous theorem proves:

$$A(C_4) < A(\Gamma) \Rightarrow C_4 < \Gamma^e.$$

Some not-too-difficult combinatorial argument $\Rightarrow C_4 < \Gamma$.

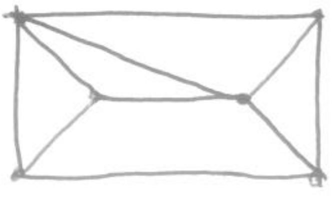
Weakly chordal conjecture

Conjecture: If $n \geq 5$ and Γ is weakly chordal then $A(C_n) \not< A(\Gamma)$.

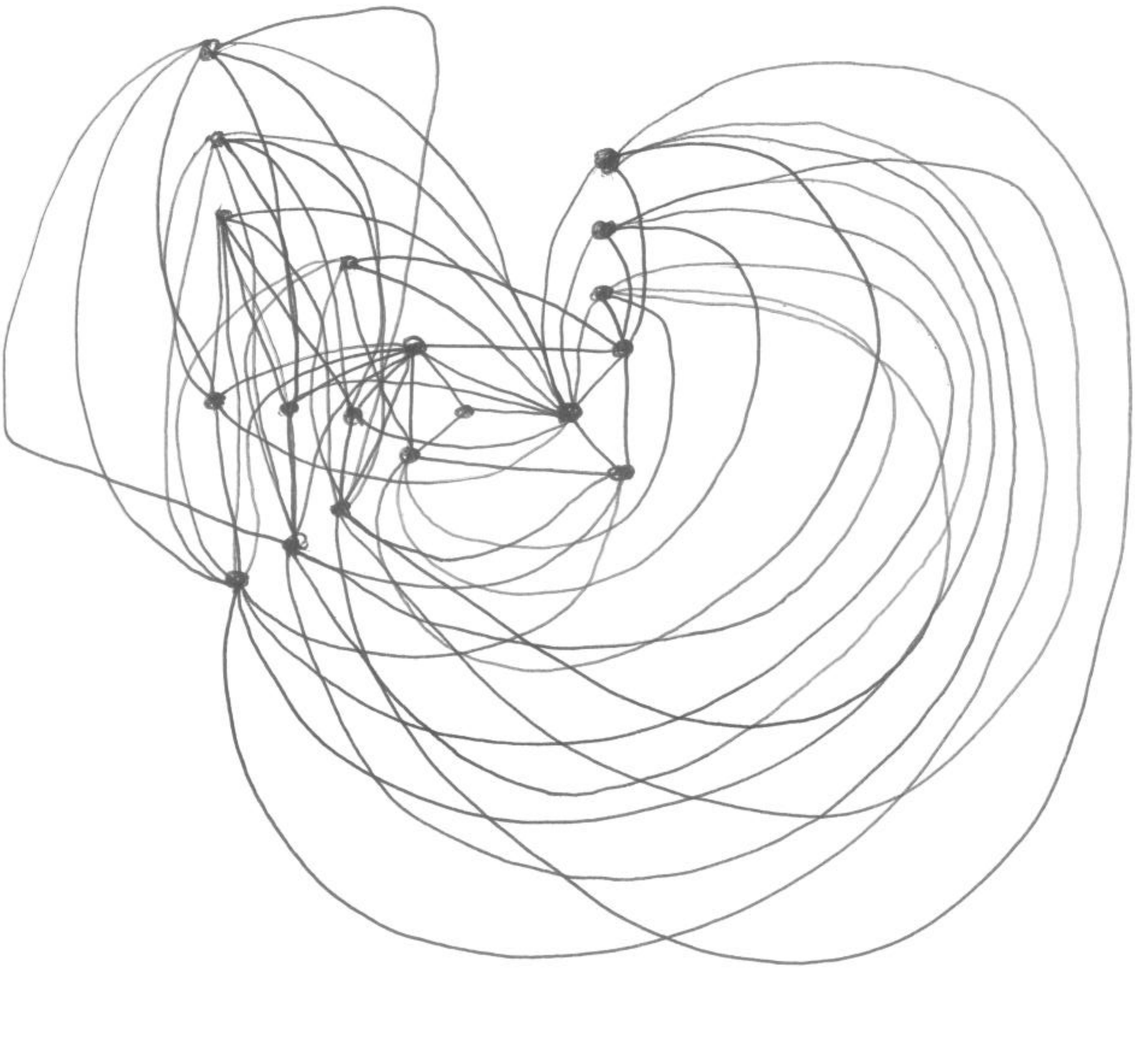
Motivation: When do we have an inclusion $\pi_1(\Sigma_g) < A(\Gamma)$, Σ_g closed, genus $g \geq 2$.

Thm (Droms): $\forall n \geq 5$ $\exists g$ s.t. $\pi_1(\Sigma_g) < A(C_n)$.

Thm (Crisp-Sageev-Sapir): $A(\overline{P_6})$ contains a closed hyperbolic surface group.

$\overline{P_6} =$ 

A very small piece of $\overline{P}_6^e$:

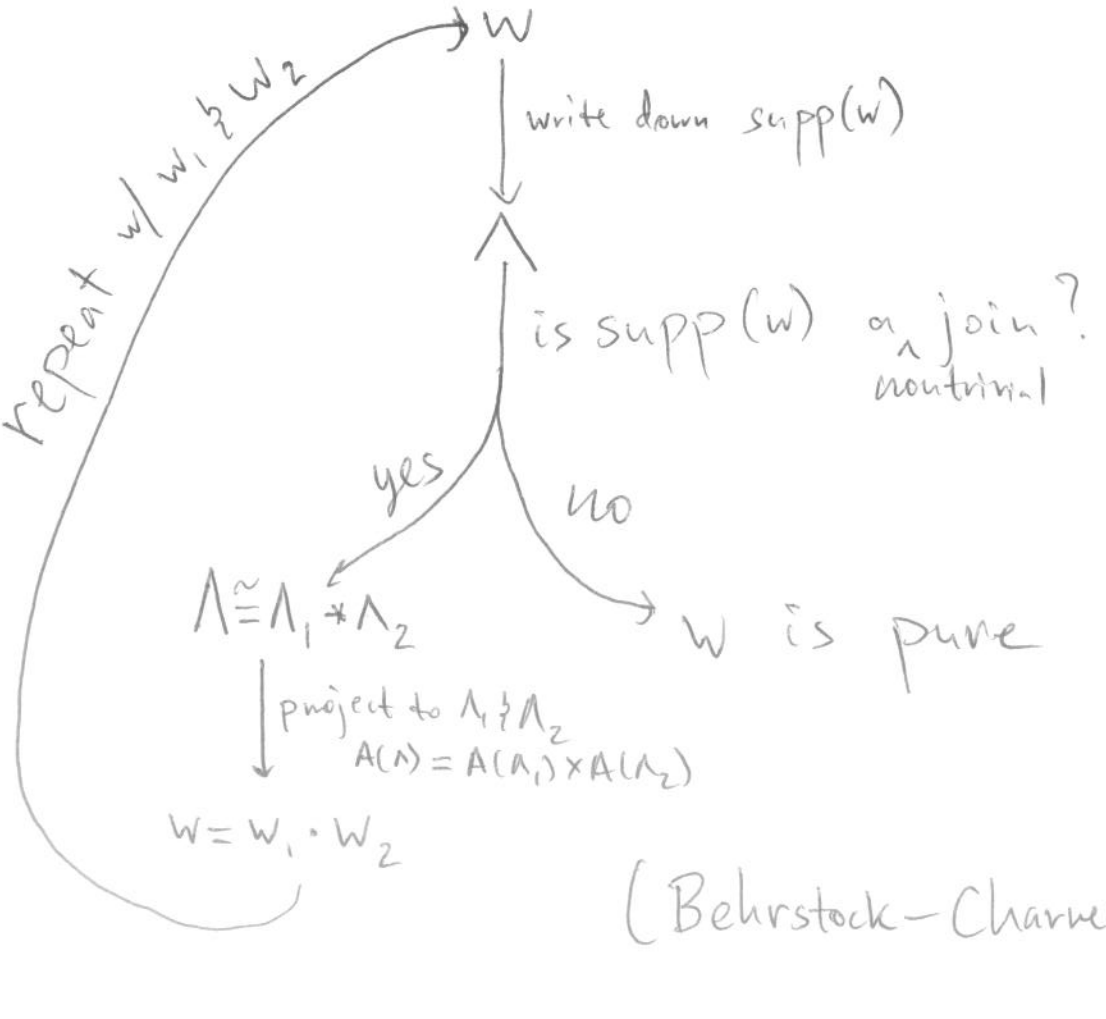


ZOMG!

So, weakly chordal conjecture + Crisp-Sapir-Sageev $\Rightarrow$ $\pi_1(\Sigma_g) < A(\Gamma)$ not arising from long cycles.

How about a proof?

Let $w \in A(\Gamma)$ be cyclically reduced.



(Behrstock-Charney)

Let $w_1, \dots, w_k \in A(\Gamma)$ be pure after cyclic reduction. $\mathcal{C}(w_1, \dots, w_k)$ is the commutation graph. $V(\mathcal{C}) = \{w_1, \dots, w_k\}$

$$E(\mathcal{C}) = \left\{ \begin{array}{r} (w_i, w_j) \text{ such that} \\ [w_i, w_j] = 1 \end{array} \right\}$$

<u>Fundamental observation</u>: $\mathcal{C} \hookrightarrow \Gamma^e$

Main points of the proof:

Choose a configuration of curves s.t. powers of Dehn twists about the curves generate $A(\Gamma) < \mathrm{Mod}(\Sigma)$.

x_i = cyclic reduction of w_i

$w_i = g_i x_i g_i^{-1}$

$\mathrm{Supp}(x_i)$ = union of curves C_i.

C_i fills a subsurface S_i.

Observe: since x_i is pure, S_i is connected.

Conjugate $\{S_i\}$ by $\{g_i\}$ to get a collection of surfaces $\{\Sigma_i\}$.

Notice: $\Sigma_i \cap \Sigma_j \neq \emptyset \iff [w_i, w_j] \neq 1$.

The curves $g_i^{-1} C_i g_i$ fill Σ_i, and certain powers of Dehn twists about each component of $g_i^{-1} C_i g_i$ are elements of $A(\Gamma) < \mathrm{Mod}(\Sigma)$. There exists a product of these powers of Dehn twists which is pseudo-Anosov on Σ_i. Choose such a pA ψ_i, and fix a component $c_i \in g_i^{-1} C_i g_i$.

For any $m \gg 0$, $\psi_i^m(c_i) \cap \psi_j^m(c_j) = \emptyset \iff \Sigma_i \cap \Sigma_j = \emptyset$.

Now, sufficiently high powers of twists about $\{\psi_i^m(c_i)\}$ are elements of $A(\Gamma)$ <u>and</u> generate a copy of $A(C)$ which arises from an inclusion

$$C \hookrightarrow \Gamma^e.$$

Cor: If $A(\Lambda) \hookrightarrow A(\Gamma)$ and

$$\underset{\text{vertex}}{\lambda_i} \longmapsto \underset{\text{pure after cyclic reduction}}{w_i}$$

then $\Lambda \hookrightarrow \Gamma^e$.

In general, images of vertices in $A(\Lambda)$ are not pure.

If Γ is any graph, Γ_k is the clique graph.

For each complete subgraph $K < \Gamma$, add a vertex v_K and:

1. join with K
2. join with all common neighbors of K.

Thm: Let Γ be any finite graph and let $A(\Lambda) < A(\Gamma)$. Then

$$\Lambda \hookrightarrow (\Gamma^e)_k .$$

Some additional remarks / results

Universal RAAGs.

A RAAG $A(\Gamma)$ with $cd\, A(\Gamma) = n$ is a universal n-dim RAAG if for each other $A(\Lambda)$ of $c.d. \leq n$, $A(\Lambda) \hookrightarrow A(\Gamma)$

cohomological dimension = size of the largest complete subgraph of Γ.

If $c.d.\, A(\Lambda) > c.d.\, A(\Gamma)$ then $A(\Lambda) \not\hookrightarrow A(\Gamma)$

c.d. 1: F_2 contains all other f.g. free groups, so it is universal, 1-dim.

c.d. 2: $cd = 2 \Leftrightarrow \Gamma$ has no $\triangle$.

Universal 2-dim RAAG does not exist. (Q. of McMullen, Sapir) indep.

pf: If Γ is n-colorable then so is Γ^e.

If $A(\Lambda) \hookrightarrow A(\Gamma)$ then Λ is n-colorable.

Erdős $\Rightarrow$ $\exists$ graphs with chromatic number arbitrarily large and girth arbitrarily large, e.g. Δ-free. $\square$

P_4-rigidity.

Thm: $A(P_4) < A(\Gamma) \iff P_4 < \Gamma$.

Easy lemma:
i. Γ is P_4-free
ii. Γ^c is P_4-free
iii. Each conn. component of an arbitrary induced subgraph of Γ is an isolated vertex or splits as a join.

pf of thm: Suppose $\phi: A(P_4) \hookrightarrow A(\Gamma)$, Γ is P_4-free. Since $A(P_4)$ is freely indec., Γ is conn. w.l.o.g. So, $\Gamma = \Gamma_1 * \Gamma_2$.

Both Γ_1 & Γ_2 are P_4-free. $\pi_i : A(\Gamma) \to A(\Gamma_i)$

By induction, $K_i = \ker \pi_i \circ \phi$ is nontrivial.

$K_1 \cap K_2 = \{1\}$ & K_1 & K_2 centralize each other.

So, $K_1 K_2 \cong K_1 \times K_2$. $1 \neq K_i \lhd A(P_4)$ and P_4 is centerless. So, $K_1 \times K_2$ contains $F_2 \times F_2$, a contradiction

$\square$