THE GEOMETRY OF RIGHT-ANGLED ARTIN GROUP SUBGROUPS OF MAPPING CLASS GROUPS

joint with MATT CLAY &
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2a

S oriented surface

e.g.  of interest: $\chi(S) < 0$

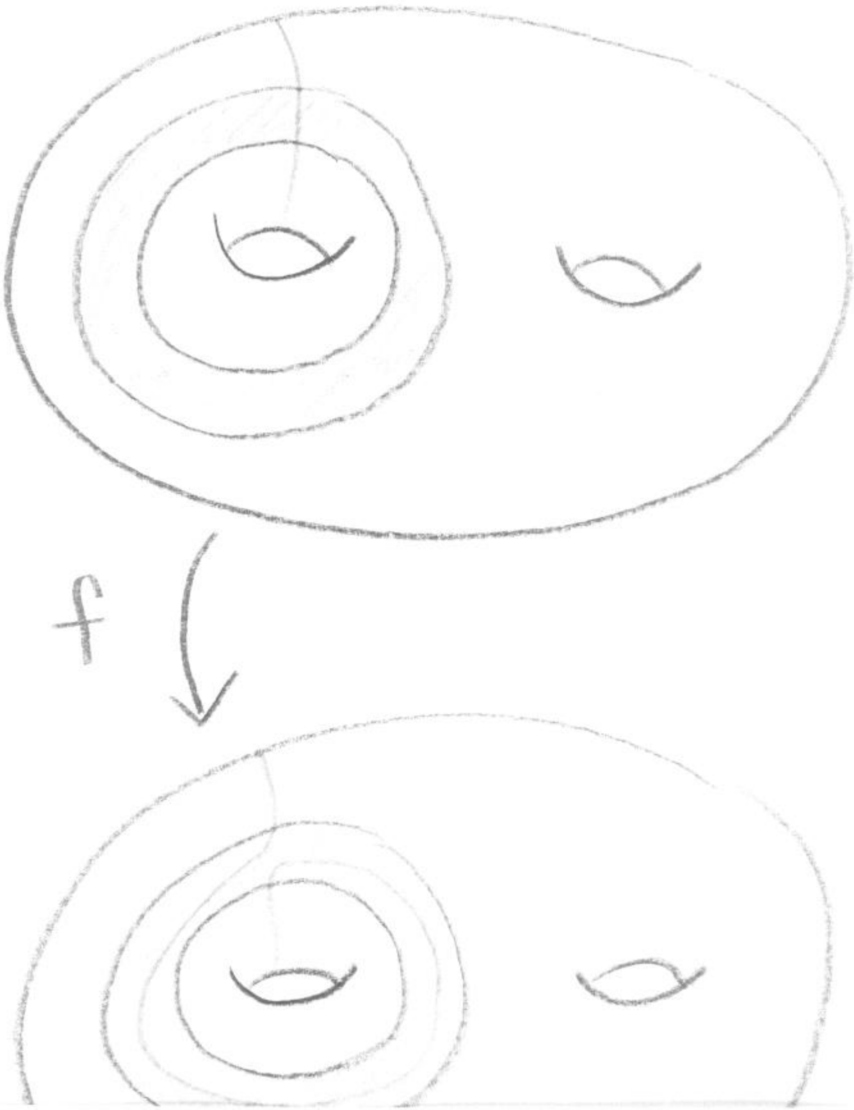
<u>*Mapping class group of S:*</u>

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Diff}^+(S))$$

$$= \left\{ f : S \xrightarrow[\text{diffeo}]{\text{o.p.}} S \right\} \Big/ \left\langle \begin{matrix} f \text{ isotopic} \\ \text{to identity} \end{matrix} \right\rangle$$

2b

Example: Dehn twist

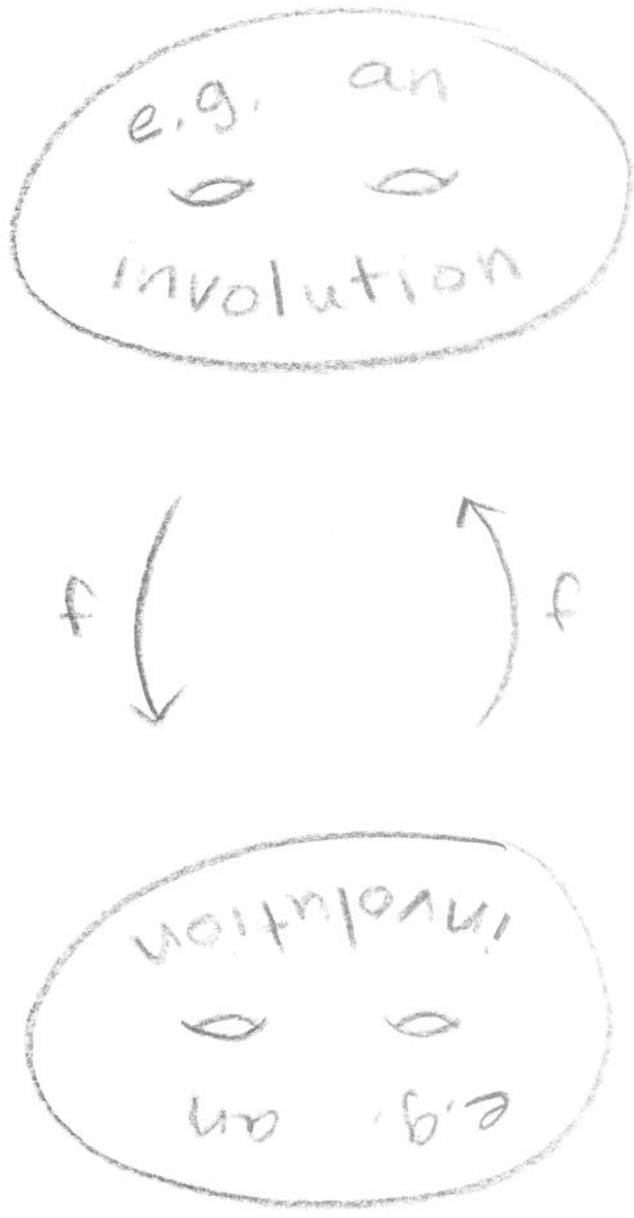


Classification (Thurston):

$f \in \mathrm{Mod}(S)$ is either

- finite-order
- pseudo-Anosov
- (infinite-order) reducible

finite order



Mod(S) $\circlearrowright$ Teich(S)

by isometries

4a

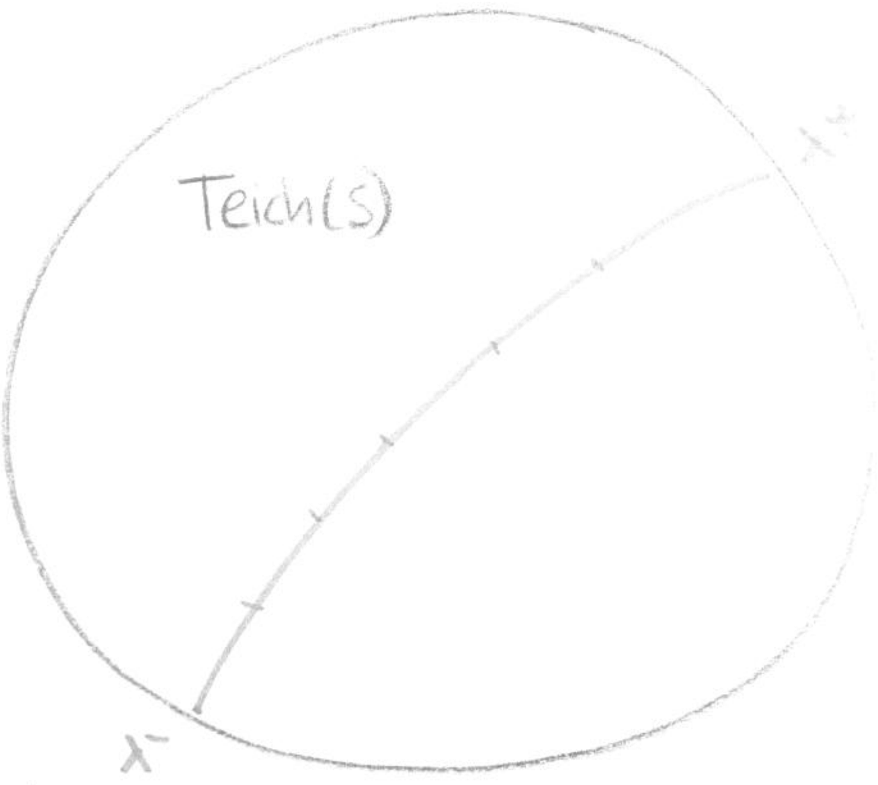
Teich(S) =

$\{(X,f)$: X is a hyp

(Riemann surface)

$f: S \to X$ homeo $\}/\sim$

A pseudo-Anosov has axis of minimal translation



$(X,f) \sim (Y,g)$ if

gf^{-1} isotopic to (conformal map)

$\mathrm{Teich}(S)/\mathrm{Mod}(S)$ = Moduli space of S

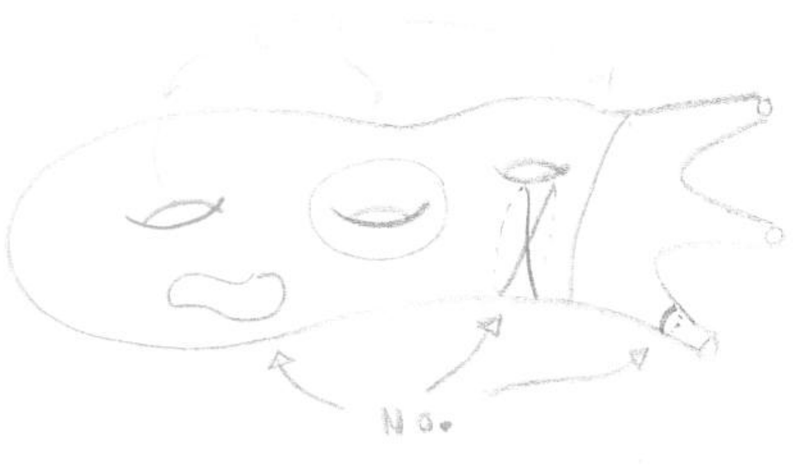
d attracting

infinite-order not pA

$\Rightarrow$ reducible:

f fixes a finite family of curves*

* isotopy classes of homotopically non-trivial non-boundary-parallel simple closed curves



Reducibles important here:

f_i pseudo-Anosov on a connected subsurface X_i

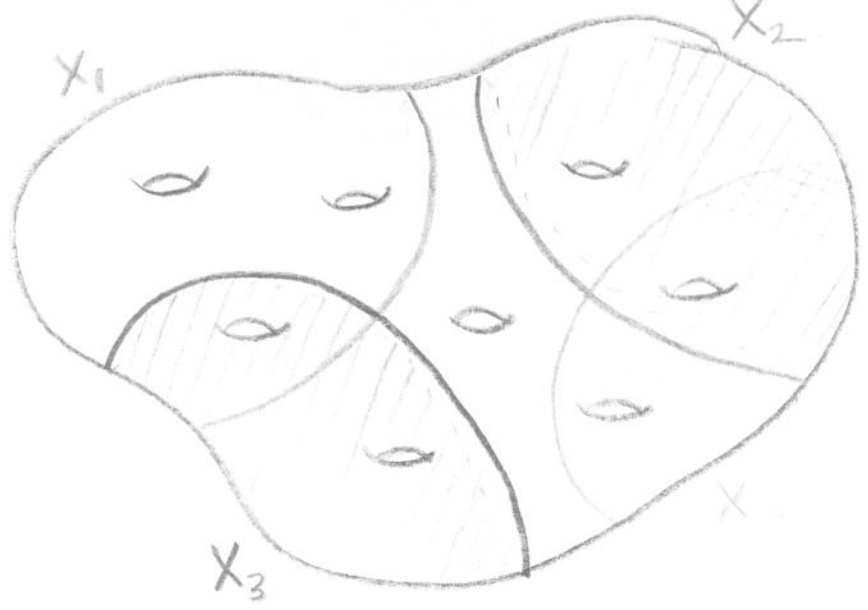
Observe: X_1 + X_2 disjoint

f_1 + f_2 commute

$\boxed{X_1, \dots X_n} \rightsquigarrow$ graph Γ

vertices $\longleftrightarrow X_i$

edges connect disjoint subsfcs.

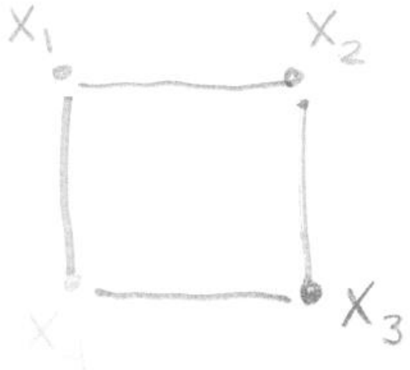


homomorphism

$A_\Gamma \longrightarrow \langle f_1, \dots f_n \rangle \subset \mathrm{Mod}(S)$

when is this

isomorphism?



<u>Theorem</u> (Crisp - Paris)

Every A_Γ embeds in some Mod(S)

<u>Theorem</u> (Koberda)

Given an irredundant set $\{f_1, \ldots, f_n\} \subset \text{Mod}(S)$, where each f_i a Dehn twist or subsurface pA supported on X_i, and $X_1, \ldots X_n \rightsquigarrow \Gamma$ "disjointness graph")

there exists N st. $\forall\, p_i > N$

$$A(\Gamma) \longrightarrow \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < \text{Mod}(S)$$

is an isomorphism.

<u>Theorem</u> (Crisp-Wiest)

Every A_Γ embeds quasi-isometrically in some $\mathrm{Mod}(S)$

<u>Theorem</u> (Clay-Leininger-M)

Given a set $\{f_1, \ldots, f_n\} \subset \mathrm{Mod}(S)$ of subsurface pAs whose support $\rightsquigarrow \Gamma$ "nicely", there exists N s.t. $\forall\, p_i > N$,

$$\begin{tikzcd} \mathbf{A}(\Gamma) \arrow[r, "①"] \arrow[dr, dashed, "②"'] \arrow[ddr, dashed, "③"'] & \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle \arrow[d] \\ & \mathrm{Mod}(S) \arrow[d] \\ & \mathrm{Teich}(S) \end{tikzcd}$$

① is isomorphism

② is quasi-isometric embedding

③ is q.i.-embedding w.r.t. standard metrics

B.5

$f: X \longrightarrow Y$ K, C-quasi-isometric embedding:

$\forall x, y \in X$

$$\frac{d_X(x,y)}{K} - C \leq d_Y(f(x), f(y)) \leq K d_X(x,y) + C$$

"$d_Y(f(x), f(y)) \prec d_X(x,y)$"

"$d_Y(f(x), f(y)) \asymp d_X(x,y)$"

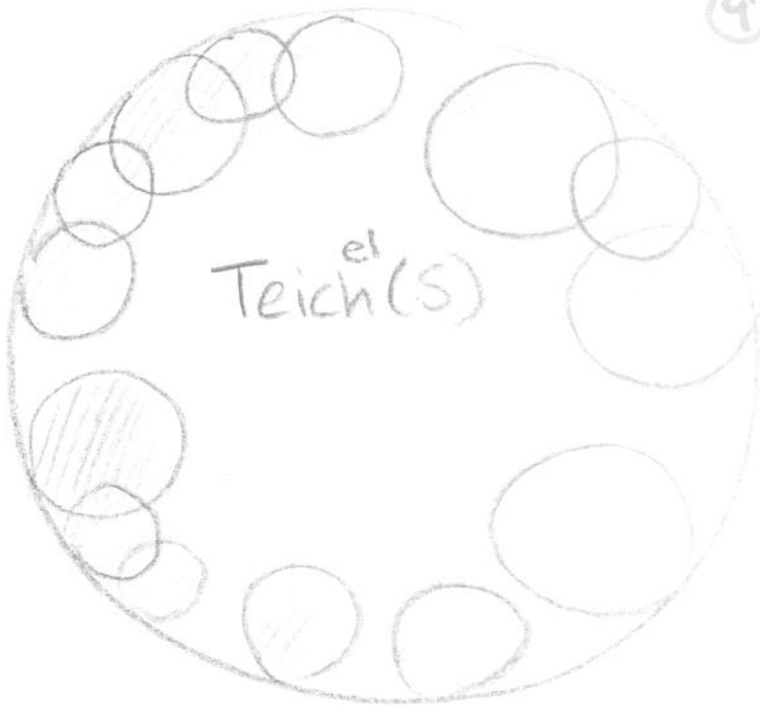
8.9

Tools in proof:

Curve complex $\mathcal{C}(S)$

vertices $\longleftrightarrow$ curves

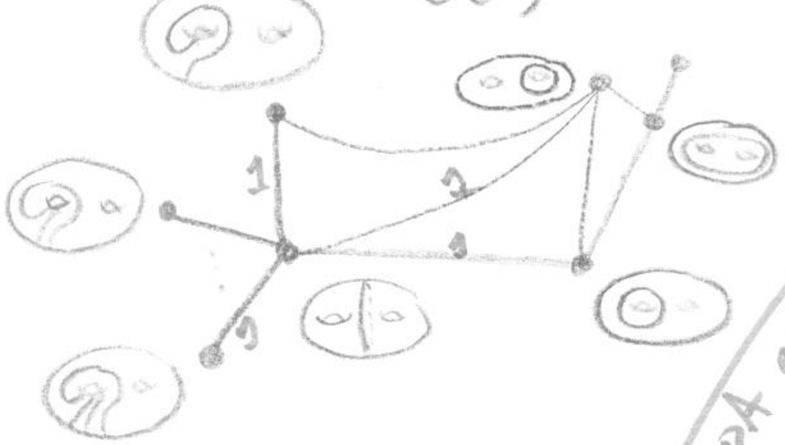
edges connect disjoint curves

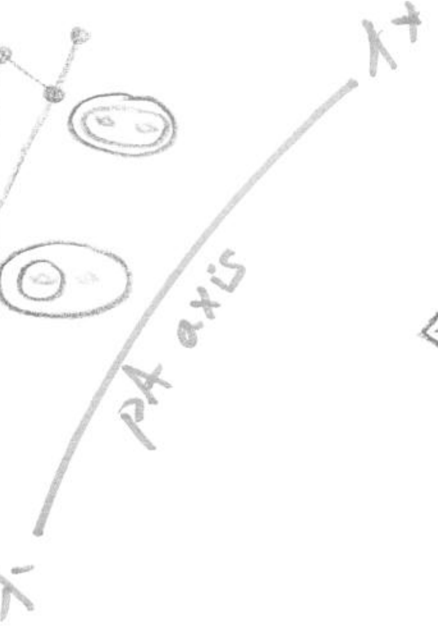


"electrified"

quasi-isometric

$\mathcal{C}(S)$



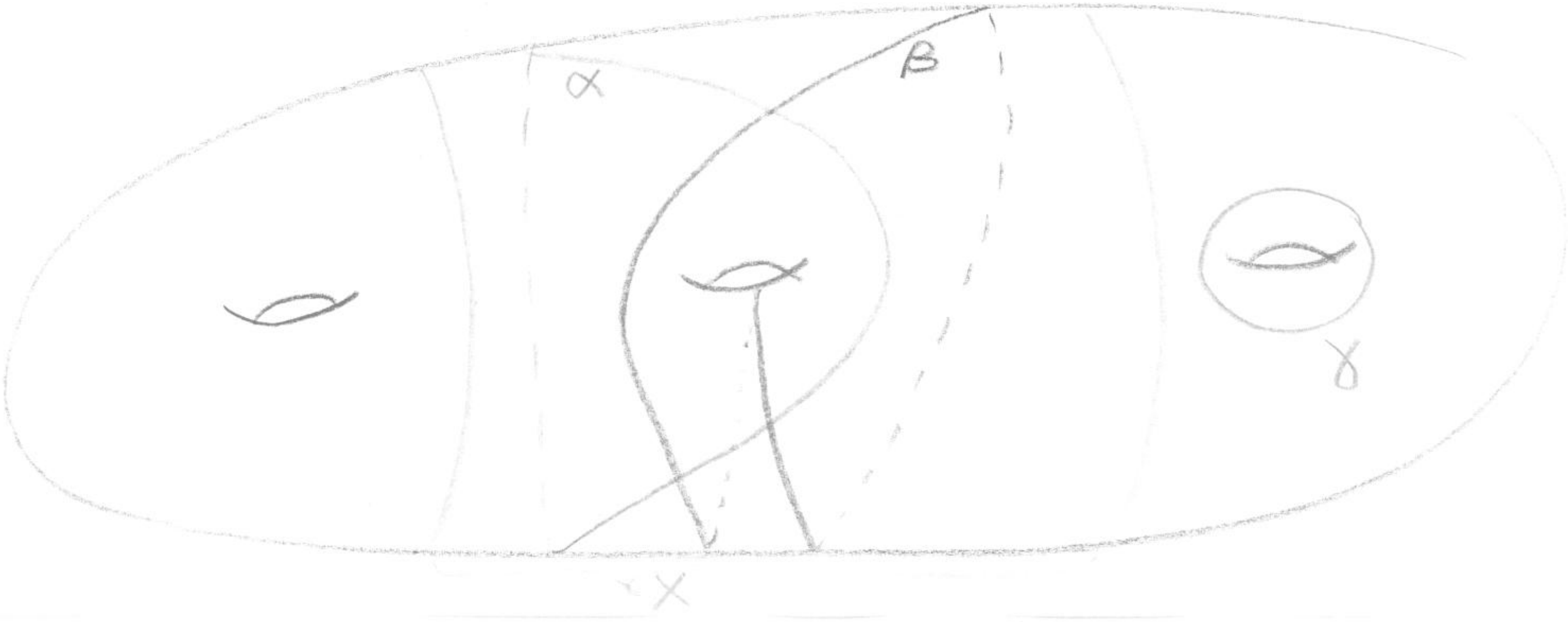


Masur-Minsky: $\mathcal{C}(S)$ is δ-hyperbolic

- locally infinite
- connected
- infinite-diameter

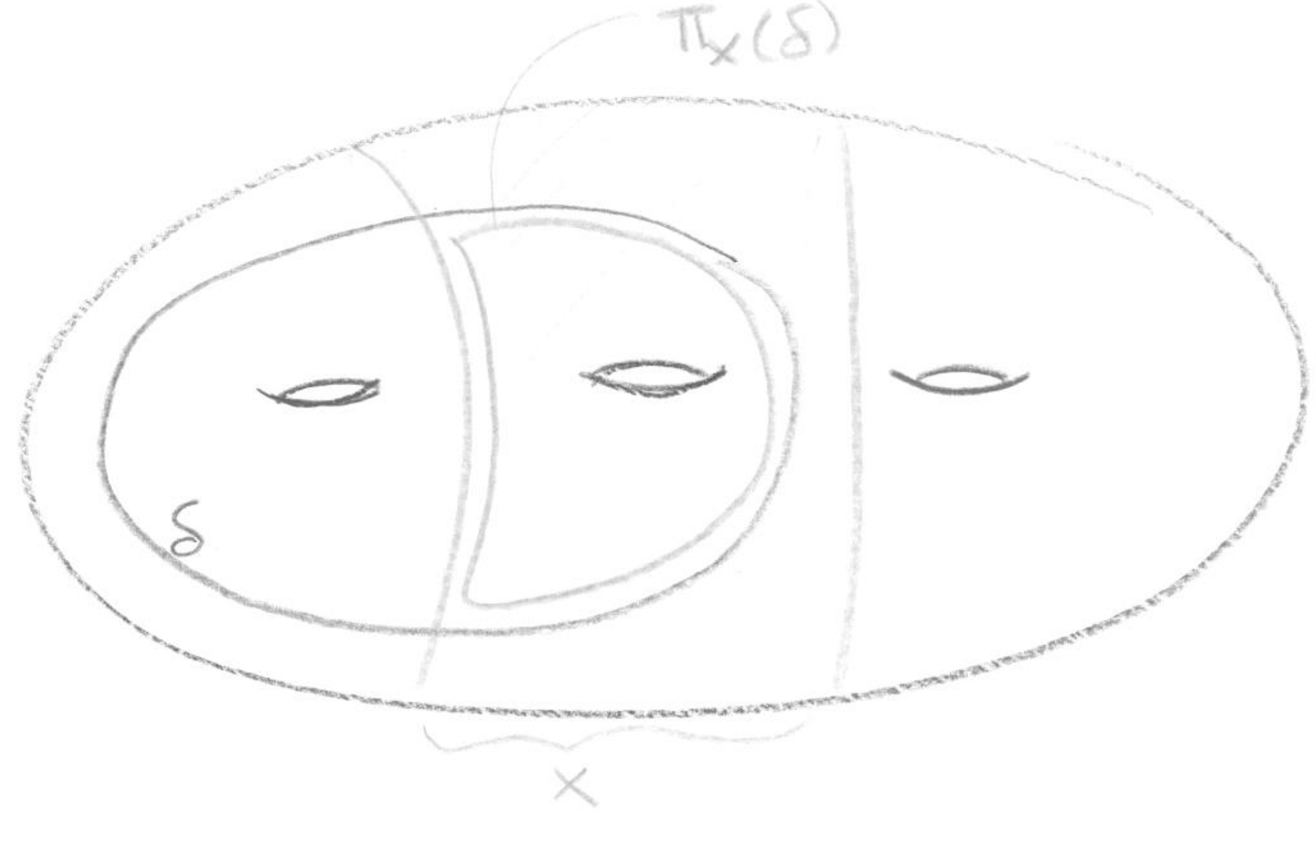
$d_{\mathcal{C}(S)}(\alpha, \beta) = 2$

$d_{\mathcal{C}(X)}(\alpha, \beta) = 3$ exercise



<u>Subsurface projection</u>:

a way to measure distances in subsurface curve complexes



Masur-Minsky formula:

Let μ be a marking. ← finite set of curves cuts S into disks (basepoint)

For large enough K,

$$|f|_{\mathrm{Mod}(S)} \overset{q.i.}{\asymp} \sum_{\substack{X \text{ subsurface of } S \\ \text{such that} \\ d_{C(X)}(\mu, f(\mu)) > K}} \left[d_{C(X)}(\mu, f(\mu)) \right]_K$$

Similar formula for Teichmüller metric (Rafi)

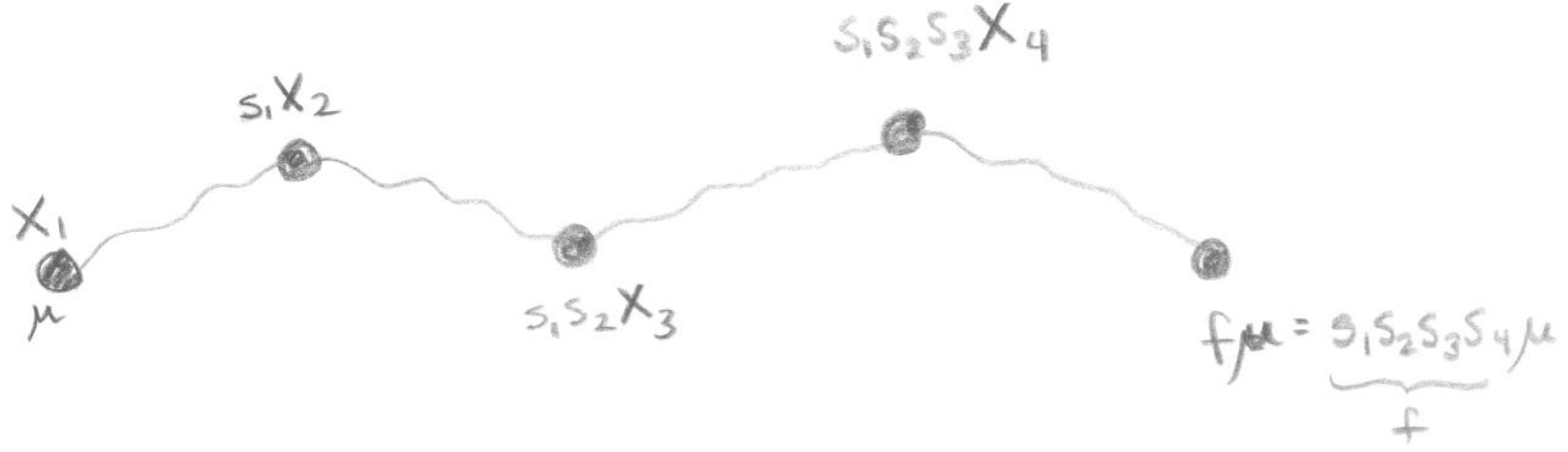
& WP metric (Brock, Masur-Minsky)

To prove $|f|_{Mod(S)} \asymp |f|_{A_\Gamma}$

EASY: $|f|_{Mod(S)} \prec |f|_{A_\Gamma}$

MAIN PART: $|f|_{A_\Gamma} \prec \sum\limits_{\substack{\text{special}\\ \text{choice}\\ X \subset S}} d_{e(X)}(\mu, f(\mu)) \leq \sum\limits_{\substack{\text{Masur-Minsky}\\ X\text{'s}}} d_{e(X)}(\mu, f(\mu))$

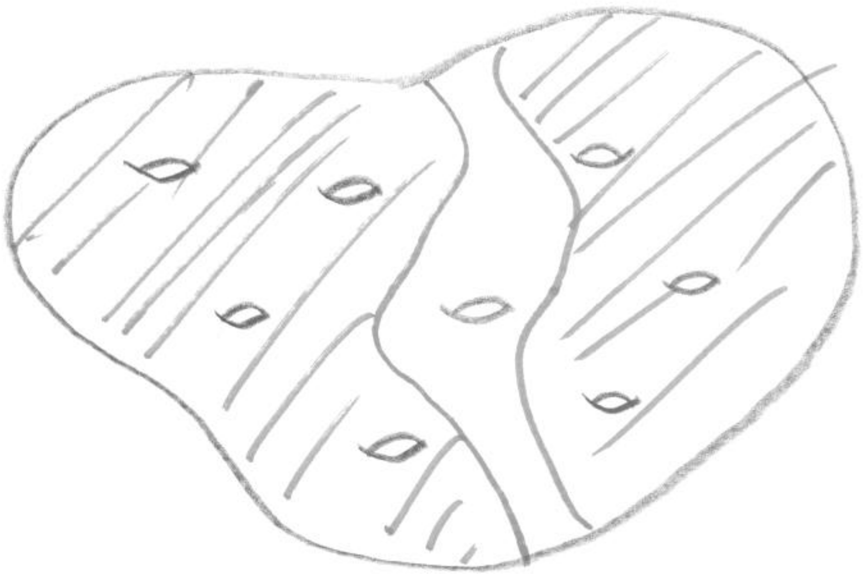
$\prec |f|_{Mod(S)}$



__Theorem__ (Clay-Leininger-M)

The word $w \in \langle f_1^{p_1}, \dots, f_n^{p_n} \rangle$ indicates the "refined Thurston type" of the mapping class w.

eg. $f_1 f_2 f_3 f_4$ is pseudo-Anosov on the subsurface:



$G < \mathrm{Mod}(S)$

convex cocompactness

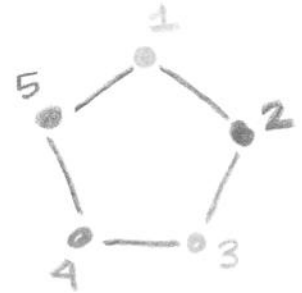
$\Leftrightarrow$ $G \xrightarrow[\text{orbit}]{} \mathcal{C}(S)$

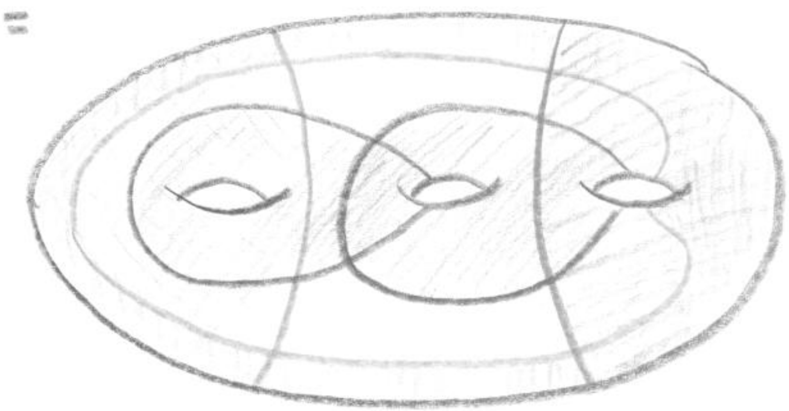
is q.i. embedding

6. All pA subgroup f.g. in $\langle f_1^{p_1}, \dots f_r^{p_r} \rangle$ is convex cocompact?

<u>Corollary</u> to first Thm (CLM)

For S with genus $g \geq 3$ & any $h \geq 2$, there are infinitely many non-conjugate genus-h surface subgroups of $\mathrm{Mod}(S)$ which act cocompactly on a qi-embedded copy of $\mathbb{H}_2$ in $\mathrm{Teich}(S)$.

<u>Proof</u> $\Gamma =$ 

$S =$ 

$\pi_1($  $) \underset{\text{[Crisp-Wiest]}}{\xhookrightarrow{\text{q.i.}}} A(\Gamma) \underset{\substack{\text{infinitely}\\\text{many:}\\\text{let } p \to \infty}}{\xhookrightarrow{\quad\quad}} \langle f_1^p, \ldots, f_5^p \rangle$