Thank the organizers!

A "Curve Complex" for the Free Group.

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n) / \mathrm{Inn}(F_n)$$

$$MCG(S_g) = \text{orient-preserv. Diffeos of } S / \text{Isotopy} \quad \left(MCG^{\pm}(S_g) = \text{all diffeos} \right.$$

closed surface.

$\cong \mathrm{Out}(\pi_1(S_g))$

Dehn-Nielsen-Baer.

If S_g is given punctures or boundary, DNB is not true...., since $\mathrm{Out}(\pi_1(S_{g,p})) = \mathrm{Out}(F_{2g})$.

(i.e. $\mathrm{Out}(F_n)$ is a generalization of certain MCG)

How to study MCG(S)?

- Action on $\mathcal{T}(S)$ = "Teichmüller space"

= deformation space of hyperbolic structures on S.

= deformation space of ~~cert~~ certain discrete actions of $\pi_1(S)$ on $\mathbb{H}^2$.

How to study Out(F_n)???

Culler-Vogtmann thought there should be something like $\mathcal{T}(F_n)$, so they invented something like that.

CV_n = "Outer Space"

= deformation space of discrete actions of F_n on cayley trees for F_n.

So.

$MCG(S_g) \curvearrowright T(S)$

free, discrete actions of $\pi_1(S)$ on $\mathbb{H}^2$

$Out(F_n) \curvearrowright CV_n$

free, discrete actions of F_n on metric trees T with $Vol(T/F_n) = 1$.

Length Spectrum

Given a marked hyperbolic surface X, get the marked length spectrum.

$$l_X : \left\{ \begin{matrix} \text{isotopy classes} \\ \text{of curves on} \\ S \end{matrix} \right\} \to \mathbb{R}.$$

$$\underbrace{[c] \longmapsto \inf_{c \in [c]} \text{length}_X(c)}$$

isotopy class

l_S determines S

Thm (9g-9 Thm) $\exists$ a set $\mathcal{L}$ 9g-9 curves s.t. $l_S|_{\{\text{this set}\}}$ determines S

$\Rightarrow$ embed $T(S) \hookrightarrow \mathbb{R}^{\mathcal{C}}$ ← can take to be finite.

Given an action of F_n on a tree get the (marked) length spectrum

$$l_T : \left\{ \begin{matrix} \text{conjugacy class} \\ \text{of elements} \\ \text{of } F_n \end{matrix} \right\} \to \mathbb{R}.$$

$$\underbrace{[g] \longmapsto \inf_{x \in T} d(x, \ldots}$$

conj. class

l_T determines T. ← (Culler–Morgan)

Thm (Smillie–Vogtmann) no finite set suffices.

$\Rightarrow$ embed $CV_n \longrightarrow \mathbb{R}^{\mathcal{C}}$ ← no finite set suffices.

$MCG(S_g) \curvearrowright T(S)$ properly discontinuously

$Out(F_n) \curvearrowright CV_n$ properly discontinuously

get (co)homological info. about $MCG(S_g)$, $Out(F_n)$.

Back to:

$T(S) \hookrightarrow \mathbb{R}^{\mathcal{C}}$

$CV_n \hookrightarrow \mathbb{R}^{\mathcal{C}}$

To mimick the criterion from $T(S)$ that each pt X has curvature -1, hence by Gauss-Bonnet, they all have the same volume, we demand that for any $(F_n \curvearrowright T) \in CV_n$, we have

$$\mathrm{vol}(T/F_n) = 1.$$

sum of the lengths of edges

$\rightsquigarrow$

$\mathcal{T}(S) \hookrightarrow \mathbb{R}^{\mathcal{C}}$

$CV_n \hookrightarrow \mathbb{P}\mathbb{R}^{\mathcal{C}}$

projectivize.

Fact: The projectivization map $\mathbb{R}^{\mathcal{C}} \to \mathbb{P}\mathbb{R}^{\mathcal{C}}$ restricts to a homeo on $\mathcal{T}(S) \subseteq \mathbb{R}^{\mathcal{C}}$.

$\vdots$

~~Thinen~~ The images.

$\mathcal{T}(S) \subseteq \mathbb{P}\mathbb{R}^{\mathcal{C}}$ $\qquad\qquad$ $CV_n \subseteq \mathbb{P}\mathbb{R}^{\mathcal{C}}$

are relatively compact.

$\overline{\mathcal{T}(S)} \subseteq \mathbb{P}\mathbb{R}^{\mathcal{C}}$ $\qquad\qquad$ $\overline{CV_n} \subseteq \mathbb{P}\mathbb{R}^{\mathcal{C}}$.

Thurston compactification

Boundaries:

$\partial \mathcal{T}(S) := \overline{\mathcal{T}(S)} \setminus \mathcal{T}(S)$

$\quad \shortparallel$

$\quad \mathcal{PML}$

"projectivized classes of measured laminations on S"

$\quad \shortparallel$ (Skora)

"projectivized classes of certain actions of $\pi_1(S)$ on $\mathbb{R}$-trees".

$\partial CV_n = \overline{CV_n} \setminus CV_n$.

"projectivized classes of actions of F_n on certain $\mathbb{R}$-trees".

What is an $\mathbb{R}$-tree?

Def (Tits). A metric space (T, d) is an $\mathbb{R}$-tree (or just tree) if $\forall x, y \in T$ there is a unique topological arc, $[x,y]$, connecting x to y in T and $[x,y]$ is isometric to $[0, d(x,y)] \subseteq \mathbb{R}$.

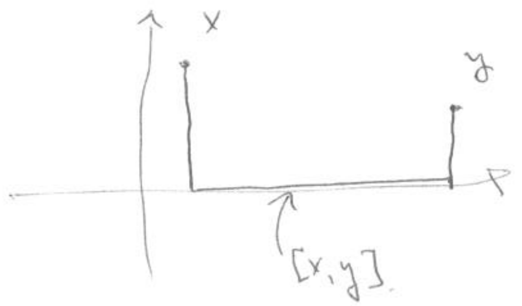
Eg:

- 

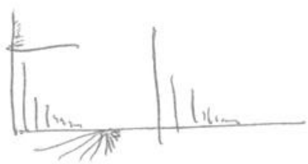
finite "simplicial" metric trees.

- 
other simplicial metric trees.

- $\mathbb{R}^2$ with the "marina metric"



(Branch points in trees need not form a discrete set.)

- 
(Compact but not finite)

more interesting things ....

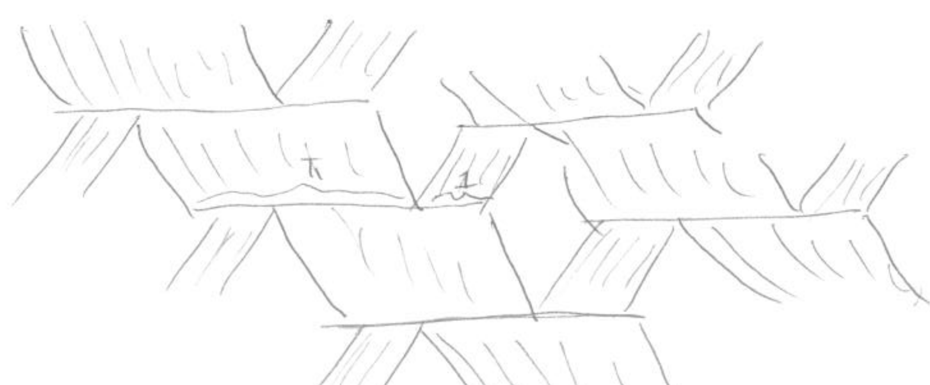
- "Surface trees".

$(X, \mathcal{F})$

(An easier representation of the foliation.)

Note: $\pi_1(\text{⊖⊖}) = F_2$, so F_2 acts on $(\tilde{X}, \tilde{\mathcal{F}})$ via deck transformations.

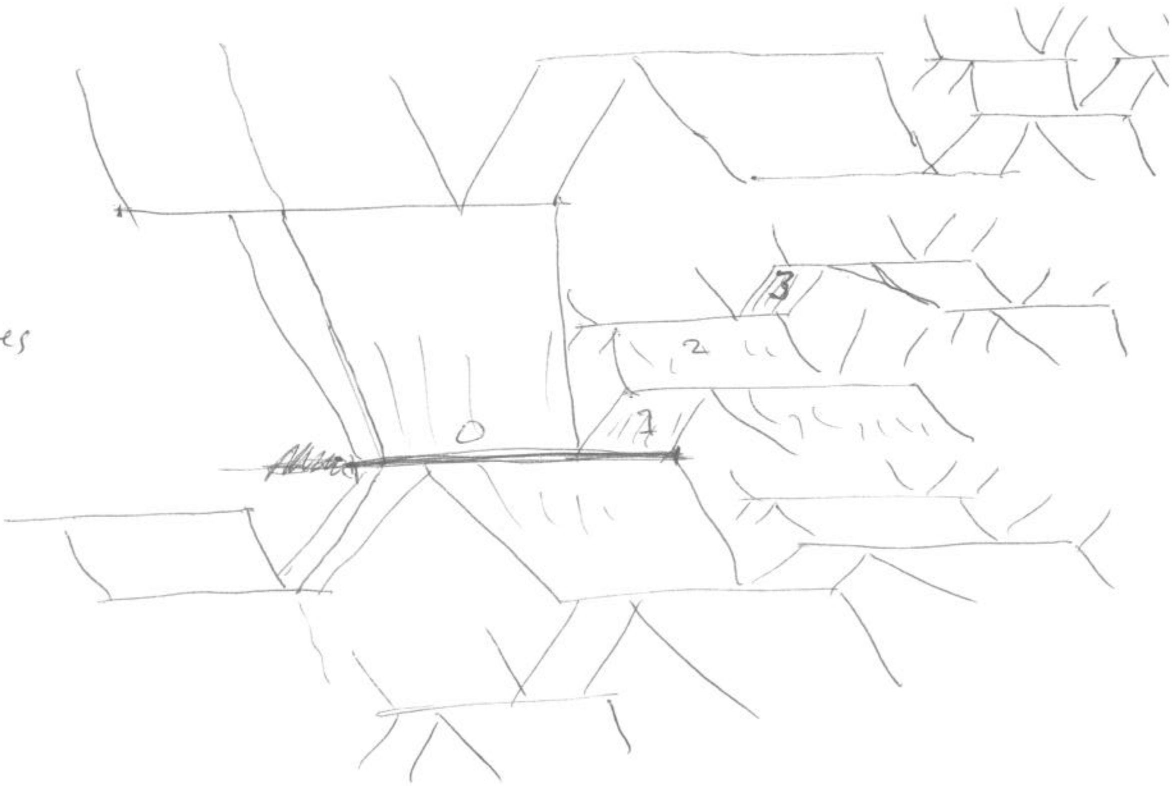
$(\tilde{X}, \tilde{\mathcal{F}})$



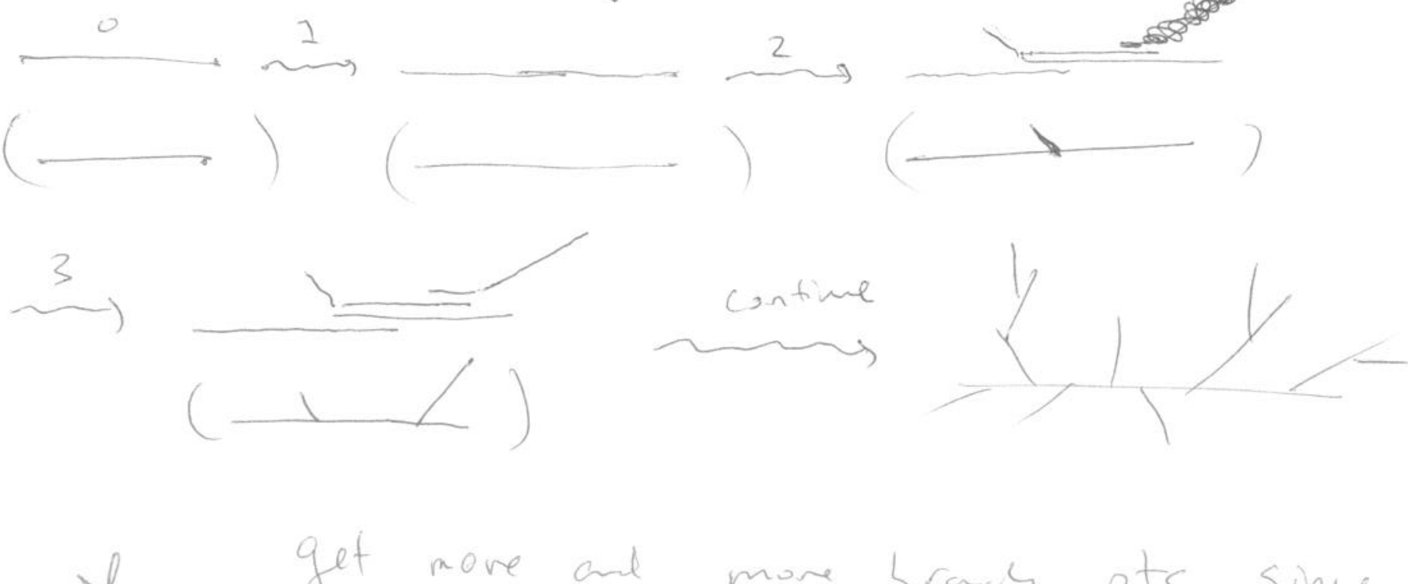
$\circlearrowleft F_2$

"surface trees"

$F_2 \curvearrowright$ action preserves $\tilde{\mathcal{F}}$ and the transverse measure.



collapse leaves



* get more and more branch pts. since $\pi, 1$ are rationally independent branch pts. are dense in the dual tree.

"Surface trees"

- Since $F_2 \curvearrowright \widetilde{X}$ preserved the transverse measure on $\widetilde{\mathcal{F}}$, $F_2 \curvearrowright \widetilde{X}$ descends to

$F_2 \curvearrowright T$ by isometries.

T is an increasing union of trees, hence T is a tree.

Fact: (Morgan-Shalen) This works for closed surfaces

.... but the picture is more difficult to draw....

Cor: $\mathcal{PML}(S)$ is a space of actions of $\pi_1(S)$ on (certain kinds of) trees. (Skora says it is the space of small actions

Fact: If S has boundary (or punctures), then $\mathcal{PML}(S)$ embeds in some ∂CV_n, so ∂CV_n generalizes $\mathcal{PML}$ in some sense.

Intersection (and more on $\mathcal{PML}$)

example of measured lamination. $(\mathcal{L}, \mu)$.

"weighted curves".

- given any curve c on S, we define.

$$i(\mathcal{L}, c) = \#\begin{pmatrix}\text{intersections of} \\ c \text{ with } c_1\end{pmatrix} \cdot 3 + \#\begin{pmatrix}\text{intersection} \\ \text{of } c \text{ with} \\ c_2\end{pmatrix} \cdot 1.$$

Thm (Thurston) ~~i(L, ...)~~ Weighted simple closed curves are dense in $\mathcal{ML}$, and $i(\cdot,\cdot)$ extends to a continuous function on $\mathcal{ML} \times \mathcal{ML} \longrightarrow \mathbb{R}_{\geq 0}$.

(In particular curves can be regarded as elements of $\mathcal{PML}$).

. . . There is an analogous picture for ~~O~~ F_n and ∂CV_n, but it is more complicated . . .

The Curve Complex:

Def (Harvey)
$\mathcal{C}(S)$ is a simplicial complex s.t.

- $(\mathcal{C}(S))^{(0)} = \{$ isotopy classes of essential simple closed curves in S $\}$.

- $(c \text{ — } c') \in (\mathcal{C}(S))^{(1)}$ iff $i(c,c') = 0$.

- $\Big($  $\Big) \in (\mathcal{C}(S))^{(2)}$ iff $i(c_i, c_j) = 0 \ \forall\, i,j$.

⋮

etc...

Def: A top dimensional simplex in $\mathcal{C}(S)$ is called a <u>pants decomposition</u> of S.

We metrize $\mathcal{C}(S)$ by identifying each simplex with the standard Euclidean simplex.

Is $\mathcal{C}(S)$ interesting?

We saw yesterday that $\mathcal{C}(S)$ codes intersection patterns of regions in $\mathcal{T}(S)$ where certain curves are short. (cf collar lemma)

<u>lemma</u>: $\mathcal{C}(S)$ has infinite diameter (in all except for a few cases...).

First.... recall that a mapping class $\alpha \in MCG(S)$ is <u>pseudo-Anosov</u> if α preserves a projective measured lamination $\mathcal{L}_\alpha$ on S.

- For such an $\mathcal{L}$ $i(\mathcal{L}, c) \neq 0$ for any curve c.

- we'll prove the ~~the~~ lemma by showing that every pseudo-anosov acts on $\mathcal{C}(S)$ with unbounded orbits.

<u>pf</u>

Choose $c \in \mathcal{C}(S)$ and suppose $\alpha^k(c)$ forms a bounded set in $\mathcal{C}(S)$. Up to subsequence $d(c, \alpha^k(c)) = r$. Choose geodesics $c = c_0^k, c_1^k, \ldots, c_r^k = \alpha^k(c)$ in $\mathcal{C}(S)$. By def., $i(c_j^k, c_{j+1}^k) = 0$. After a subsequence each $(c_j^k)_{k \in \mathbb{N}} \to \lambda^j \in PML$.

<u>Facts</u>: (1) For any curve c, $\alpha^k(c) \to \lambda_\alpha$ in PML.
(2) If $\lambda' \in ML$ with $i(\lambda', \lambda_\alpha) = 0$, then $\lambda' = \lambda_\alpha$ in PML.

This is a contradiction, since $c \in PML$ is not λ_α.

$\square$

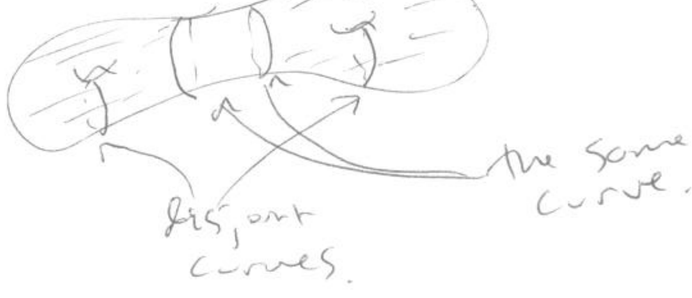
(1

Equivalent descriptions of $\mathcal{C}(S)$ (up to quasi-isometry).

$\mathcal{C}'(S) = \{$ essential connected subsurfaces of $S \}$

adjacency is disjointness (q.i. equiv. containment

can replace a curve with a regular neighborhood.



$\mathcal{C}''(S) = \{$ (small) splittings of $\pi_1(S)$ over $\mathbb{Z} \}$.

(Skora, or easier reference

SvK

$\rightsquigarrow \quad \pi_1(S) = G_1 *_{\mathbb{Z}} G_2$.

$\boxed{\text{nice 1-dim. submanifolds of } S \text{ are "dual" to nice 1-codim. submanifolds of } S.}$

(preserving a subsurface) $\Leftrightarrow$ (preserving a curve) $\Leftrightarrow$ (preserving a $\mathbb{Z}$-splitting

pseudo-Anosovs again...

TFAE: (1) $\alpha \in MCG(S)$ is pA.

(2) α preserves no subsurface

(3) α preserves no curve

(4) α preserves no splitting

"pseudo-Anosovs for $Out(F_n)$"

$\alpha \in Out(F_n)$.

(2') α preserves no (conjugacy class of) a proper free factor. F' (i.e. $F_n = F' * F''$).

not equivalent

~~(3') α preserves no primitive element (up to conjugacy)~~ ← element of a basis

(4') α preserves no $\mathbb{Z}$-splitting of F_n over a primitive edge group.

(Essentially the same as preserving a free splitting).

Def: Elements satisfying (2') are called fully irreducible (or iwip).

---- no name for 4' yet....

"Curve Complexes" for $Out(F_n)$:

(1) The free factor complex: FF_n

points are conjugacy classes of proper free factors., adjacency is inclusion.

- iwips act with unbounded orbits.
  ↳ Similar proof as $\mathcal{C}(S)$ using the more complicated intersection theory for $\overline{CV}_n$. (and a "comparison space")

- elements of type (4)'s (that are not iwip) fix a point.

(2) The splitting complex: S_n.

points are conj'gacy classes of splittings over primitive cyclic subgroups.

- iwips act with unbounded orbits
  ↳ similar proof as with pA's acting on $\mathcal{C}(S)$, but with a different more complicated intersection theory.

Thm (R) All elements of $\mathrm{Out}(F_n)$ of type (4') act with unbounded orbits on S_n. In particular, S_n is not equivariantly quasi-isometric to FF_n. (15).

Pf

Intersection theory ~~[illegible]~~ along with joint work with T. Coulbois, A. Hilion on "train track expansion of trees". Lots of technical blah-blah.

□.

Some related thing:

~~Thm (Bestvina-R) Then~~

Thms: (1) (Masur-Minsky) $\mathcal{C}(S)$ is Gromov hyperbolic

(2) (Bestvina-Feighn) FF_n " "

(3) (Handel-Mosher) S_n " ".

Thm: (Bestvina-R) We describe $\partial_\infty FF_n$. There are embeddings $\partial_\infty \mathcal{C}(S_{g,1}) \subseteq \partial_\infty FF_{2g} \subseteq \partial_\infty S_{2g}$.

So who are these (4') guys?

Eg: $\varphi \left\{\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} \mapsto \left\{\begin{matrix} b \\ acb \\ cd \\ dcd \end{matrix}\right\}$.

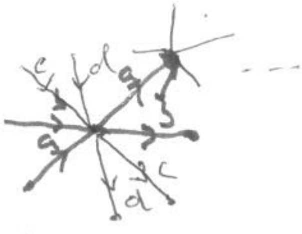
- The proper factor $\langle c,d\rangle$ is preserved.

The "limit tree" is pretty weird...

We get the limit tree T_φ by iterating φ on, say, the Cayley tree $\mathcal{C}(F_4, \{a,b,c,d\})$.

- To get convergence in $\overline{CV}_n$, we ~~rescale~~ rescale the metric as we iterate.

- Let's just consider the "axis" of **a**.



Small part of the Cayley tree.

axis of a in $\mathcal{C} =: T_0$.

- Define T_k to be T_0 where we let $g \in F_4$ act like $\varphi^k(g)$.

axis of a in T_1

$\varphi(a) = b$

$\langle a, b, c, d\rangle \mapsto \langle b, acb, cd, dcd\rangle.$

$\varphi^2(a) = acb.$

axis of a in T_2.

$\varphi^3(a) = bcdac$

axis of a in T_3.

$\langle a, b, c, d \rangle \mapsto \langle b, acb, cd, dcd \rangle$

$\varphi^4(a) = acbcddcdbcd$

<u>Observe:</u> (1) The $\langle c,d \rangle$ factor is growing faster than the elements of the $\langle a,b \rangle$ factor.

$\therefore$ In the limit the red set is very "sparse" (it is a cantor set)

(2) The action is alway color-preserving.

<u>Thm</u> (R) Any tree $T \in \overline{CV}_n$ that contains such an invariant set is (up to equivalence) a point in $\partial_\infty S_n$.