Asymptotic Geometry of $\mathcal{M}_g$

Farb-Masur: From far away, $\mathcal{M}_g$ looks like a E cone over a simplicial complex $= \mathcal{C}(S)/_{MCG(S)}$

e.g. $g=1$

$g \geq 2$.

Q: What is the geometry/topology/shape of $\mathcal{M}_g^{\varepsilon}$?

Q: What is the Teichmüller diameter of $\mathcal{M}_g^{\varepsilon}$?

Thm [Rafi-T]: $\forall \varepsilon \leq \varepsilon_m$ (Margulis constant).

$$\operatorname{diam}_T(\mathcal{M}_g^{\varepsilon}) \asymp \log(g/\varepsilon)$$

$$\left\{ \frac{1}{K} \log(g/\varepsilon) \leq \operatorname{diam}_T(\mathcal{M}_g^{\varepsilon}) \leq K \log(g/\varepsilon) \right.$$

K independent of g & ε.

<u>defn:</u> Margulis constant ε_m = genus independent constant s.t

ε_m $\forall x \in M_g$ 2 curves of length $\leq \varepsilon_m$ are disjoint

$\boxed{\text{Set } \varepsilon_m = 1.}$

~~Refinement of~~

<u>Refinement</u>:

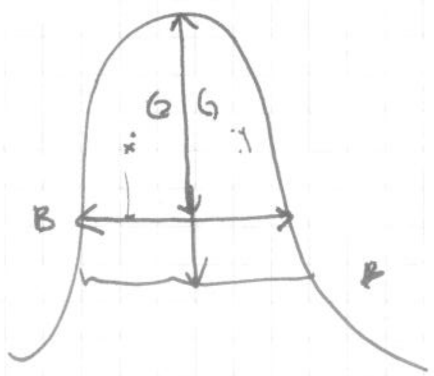
<u>defn</u>: $B \subset M_g^\varepsilon$

$$B = \left\{ X \in M_g^\varepsilon : \begin{array}{l} X \text{ admits a } \overset{\text{(unique)}}{\text{pants decomposition}}\ P \\ \text{s.t } l_X(\alpha) = 1 \quad \forall \alpha \in P \end{array} \right\}$$

<u>defn</u>:

$$\text{Width of } M_g^\varepsilon = \operatorname{diam}_T(B) \overset{\text{Thm.}}{\asymp} \log(g) \qquad \text{Combinatorial}$$

$$\text{Height of } M_g^\varepsilon = \sup_{X \in M_g^\varepsilon} d_T(X, B) \asymp \log(g/\varepsilon) \qquad \text{Geometric.}$$



Width + height $\Rightarrow$ diameter.

Lower bound arguments.

- Easier than Upper bounds.
- are obtained by examples.
- Fact:

$$d_T(X,Y) \geq \inf_f \{ \log L_f : f : X \to Y \ L_f\text{-Lipschitz} \}.$$

$$\geq \log (\text{stretch factor})$$ ~~ratios of lengths of curves~~

Height of $\mathcal{M}_g^{\varepsilon} = \sup_{X \in \mathcal{M}_g^{\varepsilon}} d_T(X, \mathcal{B}) \overset{\cup}{\sim} \log(\delta/\varepsilon)$

Recall: $\mathcal{B} = \{ X \in \mathcal{M}_g^{\varepsilon} \quad X$ has a pants decom P, $l_X(\alpha) = 1 \quad \forall \alpha \in P \}$.

- If X has a curve of length ε, then $d_T(X, \mathcal{B}) \geq \log(1/\varepsilon)$

Bers Constant

B_g = (metric ind) constant s.t $\forall X \in \mathcal{M}_g$.

$\exists$ Pants dec on P s.t $l_X(\alpha) \leq B_g \quad \forall \alpha \in P$.

Bers: $\underbrace{\sqrt{g} < B_g}$ $< g$.

- obtained by hairy torus.
- but hairy torus is Not thick.

Claim: $\exists X \in \mathcal{M}_g^{\varepsilon}$ which obtains $\sqrt{g}$ lower bound for B_g.

Modification doubling of Buser's hairy torus.

Upper bound on height

$$\sup_{X \in \mathcal{M}_g^{\varepsilon}} d_T(X, \mathcal{B}) \prec \log(g/\varepsilon)$$

Idea: For any $X \in \mathcal{M}_g^{\varepsilon}$, choose Buser's Pants decomposition P_X on X.

$$\varepsilon \leq l_X(\alpha) \prec g \quad \forall \alpha \in P_X$$

- Choose $Y \in \mathcal{B}$ where P_Y combinatorially the same as P_X.
- Show $d_T(X, Y) \prec \log(g/\varepsilon)$

"Pinch" P_X down to length 1.

Problem: a curve of length g ~~can~~ comes e^{-g} close to itself

Naive approach may not work.

- Actual proof uses extremal length estimates.

Width of $\mathcal{M}_g^{\varepsilon}$ = $\mathrm{diam}_T(\mathcal{B}) \asymp \log g$. | $\mathcal{B} \longrightarrow \mathrm{Cubic}(g)$

Lower bound: Find $X, Y \in \mathcal{B}$ so $d_T(X, Y) \succ \log g$.

Upper bound:

$\mathcal{B} \longrightarrow$ Trivalent graphs of rank g $\longrightarrow$ Tree(n) rootes tree with $n = 2g-2$ labels

$X \longmapsto$ dual graph to pants decom on X $\longrightarrow$ cut & label.

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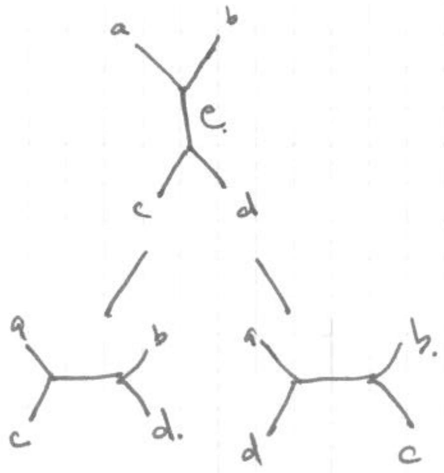
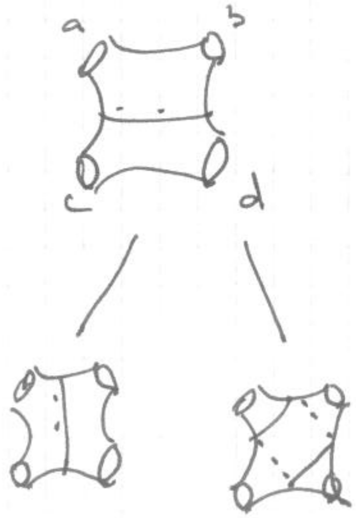
- Equip Tree(n) w/ right metric so
  $\mathrm{diam}_T(B) \asymp \mathrm{diam}(\mathrm{Tree}(n))$

Whitehead Moves:





$d_W(T, T') = 1 \iff T$ & T' differs by one WH move

d_W Not the right metric b.c. d_T is a sup metric.

Simultaneous Wh Moves

One Sim WH move = Composition of WH moves whose support are ~~do~~ pairwise joint edges

$d_S(T, T') = 1 \iff T$ & T' differ by one sim WH move.

$\mathrm{diam}_T(B) \asymp \mathrm{diam}_S(\mathrm{Tree}(n)) \quad (n = 2g-2)$.

Thm: $\mathrm{diam}_S(\mathrm{Tree}(n)) \asymp \log(n)$.

Idea of Proof :

- First transform any tree $T \in \mathrm{Tree}(n)$ to have height $\log(n)$ after $\log(n)$ moves.
- Then sort labels. to be fully-sorted ~~in $\log(n)$ moves~~ moves. Require 2 Ht ~~#~~ of moves, so $2\log(n)$ moves.

Application to Outer Space

$\overline{X}(n)$ = Outer Space / $\mathrm{Out}(\mathbb{F}_n)$ = moduli space of metric graphs of rank n of volume n.

$\mathrm{diam}_L(X_n^\varepsilon) \asymp \log(n/\varepsilon)$.

Back to Non-sim Wt Moves.

$\mathrm{diam}_W(\mathrm{Tree}(n)) \asymp \log n$.

- Upper bound is easy (consequence of ~~sim~~ diameter)
- lower bound = Bollobás + Sleator-Tarjan-Thurston
