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ON SUBMANIFOLDS AND QUOTIENTS OF POISSON AND JACOBI MANIFOLDS

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This work is dedicated to the memory of Stanisław Zakrzewski

Abstract. We obtain conditions under which a submanifold of a Poisson manifold has an induced Poisson structure, which encompass both the Poisson submanifolds of A. Weinstein [21] and the Poisson structures on the phase space of a mechanical system with kinematic constraints of Van der Schaft and Maschke [20]. Generalizations of these results for submanifolds of a Jacobi manifold are briefly sketched.

1. Introduction. Let (P, Λ) be a Poisson manifold, and D be a submanifold of P. In [21], A. Weinstein indicates two different cases in which there exists on D a Poisson manifold structure naturally induced by the Poisson structure of P. Let us recall these two cases.

First case. The submanifold D of P is called a *Poisson submanifold of the first kind* if, for each $x \in D$,

$$T_xD \supset C_x = \Lambda^\sharp(T_x^*P).$$

We have denoted by $\Lambda^\sharp : T^*P \to TP$ the vector bundle map associated with the Poisson tensor Λ, defined by

$$\langle \beta, \Lambda^\sharp \alpha \rangle = \Lambda(\alpha, \beta),$$

where α and β are two elements of T^*P which belong to the same fibre. The Poisson tensor Λ_D of D is such that, for each $x \in D$, and each η and $\zeta \in T_x^*D$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\widehat{\eta}, \widehat{\zeta}),$$

where $\widehat{\eta}$ and $\widehat{\zeta}$ are elements of T_x^*P, *i.e.*, linear forms on T_xP whose restrictions to the subspace T_xD of T_xP are equal to η and to ζ, respectively. That definition makes sense since $\Lambda(\widehat{\eta}, \widehat{\zeta})$ depends only on the restrictions of $\widehat{\eta}$ and $\widehat{\zeta}$ to T_xD.

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Geometrically, a Poisson submanifold of the first kind of (P, Λ) is a submanifold whose intersection with each symplectic leaf of (P, Λ) is an open subset of that leaf.

Second case. The submanifold D of P is called a *Poisson submanifold of the second kind* if, for each $x \in D$,

$$T_xP = T_xD \oplus \Lambda^\sharp(T_xD^0),$$

where T_xD^0 is the annihilator of T_xD, *i.e.*, the set of elements $\alpha \in T_x^*P$ such that $\langle \alpha, v\rangle = 0$ for all $v \in T_xD$. For each $\eta \in T_x^*D$, let $\widehat{\eta}$ be the unique element of T_x^*P such that

$$\langle \widehat{\eta}, v\rangle = \begin{cases} \langle \eta, v\rangle & \text{if } v \in T_xD, \\ 0 & \text{if } v \in \Lambda^\sharp(T_xD^0). \end{cases}$$

The Poisson tensor Λ_D of D is such that, for each $x \in D$, η and $\zeta \in T_x^*D$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\widehat{\eta}, \widehat{\zeta}).$$

Geometrically, a Poisson submanifold of the second kind of (P, Λ) is a submanifold of D such that, for each symplectic leaf S of (P, Λ), the intersection $D \cap S$ is everywhere transverse and is a symplectic submanifold of S.

Van der Schaft and Maschke [20] have shown that in the geometric theory of mechanical systems with kinematic constraints, it is natural to introduce, on the submanifold of the phase space of the system which corresponds to the constraint, a pseudo-Poisson tensor (*i.e.*, a two times contravariant skew-symmetric tensor), and that in some cases which belong neither to the first, nor to the second case described above, that pseudo-Poisson tensor is in fact a true Poisson tensor. Let us recall their main results.

Let N be a smooth manifold (the configuration manifold of the mechanical system), $L : TN \to \mathbf{R}$ a smooth function (the Lagrangian of the system) and C a (maybe nonintegrable) vector sub-bundle of TN (the kinematic constraint). Let $\mathcal{L} : TN \to T^*N$ be the Legendre transformation associated with the Lagrangian L. We assume that L is regular, *i.e.*, that $\mathcal{L}$ is a diffeomorphism. Let $H = \big(i(Z)dL - L\big) \circ \mathcal{L}^{-1}$ be the Hamiltonian (we have denoted by Z the Liouville vector field on TN). Then $D = \mathcal{L}(C)$ is a submanifold of T^*N. Let Λ be the (nondegenerate) Poisson tensor on T^*N associated with its canonical symplectic structure $d\alpha_N$ (we have denoted by α_N the Liouville 1-form on T^*N).

In [17], we introduced a vector sub-bundle W of the tangent bundle $T(T^*N)$, called the *projection bundle*, which can be defined by the following procedure (equivalent to that used in [17]). Let C^0 be the annihilator of C. It is a vector sub-bundle of T^*N. Let q^*C^0 be its pull-back by the canonical projection $q : T^*N \to N$. We see that q^*C^0 is a vector sub-bundle of the cotangent bundle $T^*(T^*N)$, generated by the pullbacks $q^*\xi$ of all the sections ξ of C^0, *i.e.*, of all the 1-forms ξ on N which vanish on the sub-bundle C of TN. The projection bundle W is defined as

$$W = \Lambda^\sharp(q^*C^0).$$

We assume that L is a classical Lagrangian, *i.e.*, that for each $x \in N$, the restriction of L to T_xN is a positive definite quadratic form. Then it is easy to prove (see for example [17]) that for each $z \in D$, $T_z(T^*N)$ splits into a direct sum,

$$T_z(T^*N) = T_zD \oplus W_z,$$

where W_z is the fibre at z of the vector sub-bundle W. Therefore, for each $z \in D$ and each $\eta \in T_z^*D$, there is a unique $\widehat{\eta} \in T_z^*(T^*N)$ such that

$$\langle \widehat{\eta}, v \rangle = \begin{cases} \langle \eta, v \rangle & \text{if } v \in T_zD, \\ 0 & \text{if } v \in W_z. \end{cases}$$

We can then define a two times contravariant skew-symmetric tensor Λ_D on D by setting, for each $z \in D$, η and $\zeta \in T_z^*D$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\widehat{\eta}, \widehat{\zeta}).$$

The tensor Λ_D is the pseudo-Poisson tensor on the submanifold D of T^*N, introduced by Van der Schaft and Maschke. Their main result is the following theorem.

THEOREM 1 (Van der Schaft and Maschke [20]). *The tensor Λ_D is a Poisson tensor (i.e., the bracket of functions defined by means of that tensor satisfies the Jacobi identity) if and only if the vector sub-bundle C of TN is completely integrable (i.e., involutive).*

Several researchers (Koon and Marsden [10], Cantrijn, de León and Martín de Diego [1]) have used the pseudo-Poisson tensor of Van der Schaft and Maschke for the study of mechanical systems with nonholonomic kinematic constraints. When C is completely integrable, the above theorem yields an example of a Poisson submanifold D of the (nondegenerate) Poisson manifold (T^*N, Λ) which does not belong to the first kind, nor to the second kind of Poisson submanifolds described above. Indeed, it does not belong to the first kind since $\Lambda^\sharp\big(T_z^*(T^*N)\big) = T_z(T^*N)$ is not contained in T_zD; and it does not belong to the second kind, since in general W_z is not equal to $\Lambda^\sharp(T_zD^0)$.

We shall see in Section 2 that Van der Schaft and Maschke's Theorem 1 is closely related to a result due to P. Libermann [12] about the quotient of a symplectic manifold by a suitable foliation. In Section 3, we will generalize the result of P. Libermann for quotients of Poisson manifolds. Then in Section 4 we will obtain (Lemmas 1 and 2 and Proposition 2) conditions under which a submanifold of a Poisson manifold has a Poisson structure, which include the two kinds of Poisson submanifolds described by Weinstein, as well as the new kinds of Poisson submanifolds obtained by application of the Van der Schaft and Maschke's theorem. Finally, in Section 5, we will look at what happens when we replace Poisson manifolds by Jacobi manifolds.

2. Foliated symplectic manifolds. Let (M, Ω) be a symplectic manifold and $\mathcal{F}$ a completely integrable sub-bundle of TM. We assume that the set of leaves $M/\mathcal{F}$ of the foliation defined by $\mathcal{F}$ has a smooth manifold structure and that the canonical projection $\pi : M \to M/\mathcal{F}$ is a submersion. Let us recall the theorem:

THEOREM 2 (P. Libermann [12]). *Let* $\operatorname{orth}\mathcal{F}$ *be the symplectic orthogonal of the vector sub-bundle $\mathcal{F}$. The three properties below are equivalent:*

(i) *the vector bundle* $\operatorname{orth}\mathcal{F}$ *is completely integrable (involutive);*

(ii) *for every pair (f, g) of smooth functions, defined on an open subset of M, whose differentials df and dg vanish on $\mathcal{F}$, the differential $d\{f, g\}$ of their Poisson bracket vanishes on $\mathcal{F}$;*

(iii) *the quotient manifold $M/\mathcal{F}$ has a Poisson structure for which the projection $\pi : M \to M/\mathcal{F}$ is a Poisson map.*

When these equivalent properties are satisfied, the Poisson structure on $M/\mathcal{F}$ for which Property (iii) *is satisfied is unique.*

Let us briefly sketch the proof. The vector bundle map $\Lambda^\sharp : T^*M \to TM$ is a vector bundle isomorphism which maps the annihilator $\mathcal{F}^0$ of $\mathcal{F}$ onto $\operatorname{orth}\mathcal{F}$ (we have denoted by Λ the nondegenerate Poisson tensor associated with the symplectic form Ω). Therefore, $\mathcal{F}^0$ is generated by the differentials df, and $\operatorname{orth}\mathcal{F}$ by the Hamiltonian vector fields $\Lambda^\sharp(df)$, for all smooth functions f on open subsets of M whose differentials vanish on $\mathcal{F}$. The results follow directly from the fact that, for every pair (f, g) of smooth functions defined on an open subset of M, we have $\big[\Lambda^\sharp(df), \Lambda^\sharp(dg)\big] = \Lambda^\sharp\big(d\{f, g\}\big)$.

Remark. Under the assumptions of Theorem 2, when both $M/\mathcal{F}$ and $M/\operatorname{orth}\mathcal{F}$ have smooth manifold structures for which the canonical projections $M \to M/\mathcal{F}$ and $M \to M/\operatorname{orth}\mathcal{F}$ are submersions, these two quotient manifolds, equipped with the Poisson structures for which the canonical projections are Poisson maps, make a *dual pair* in the sense of A. Weinstein [21]. Properties of dual pairs are thoroughly discussed by M.V. Karasev and V.P. Maslov in their book [7], which also contains many new results about Poisson manifolds, symplectic groupoids and applications to quantization. Dual pairs were considered earlier by C. Carathéodory [2], under the name of *function groups*, *polar* of each other.

Application. Let us now explain why Van der Schaft and Maschke's Theorem 1 is a direct consequence of Libermann's Theorem 2. With the notations of Theorem 1, let us first prove that the projection bundle W is always an isotropic, completely integrable vector sub-bundle of $T(T^*N)$. It is generated by vector fields on T^*N of the type $\Lambda^\sharp(q^*\eta)$, where η is a section of C^0, *i.e.*, a 1-form on N which vanishes on C. Let $(x^1, \dots, x^n)$ be a system of local coordinates on N and $(x^1, \dots, x^n, p_1, \dots, p_n)$ be the correspondig system of canonical local coordinates on T^*N. We have, locally,

$$\eta = \sum_{i=1}^n \eta_i(x^1, \dots, x^n)dx^i, \quad \Lambda = \sum_{i=1}^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i},$$

therefore

$$\Lambda^\sharp(q^*\eta) = -\sum_{i=1}^n \eta_i(x^1, \dots, x^n)\frac{\partial}{\partial p_i},$$

and we see that W is isotropic. Let us now consider two sections η and ζ of C^0, and the corresponding vector fields $\Lambda^\sharp(q^*\eta)$ and $\Lambda^\sharp(q^*\zeta)$. Their local expressions are

$$\Lambda^\sharp(q^*\eta) = -\sum_{i=1}^n \eta_i(x^1, \dots, x^n)\frac{\partial}{\partial p_i}, \qquad \Lambda^\sharp(q^*\zeta) = -\sum_{i=1}^n \zeta_i(x^1, \dots, x^n)\frac{\partial}{\partial p_i},$$

and therefore $\big[\Lambda^\sharp(q^*\eta), \Lambda^\sharp(q^*\zeta)\big] = 0$, since the η_i and ζ_i do not depend on the coordinates $p_1, \dots, p_n$. This shows that W is completely integrable. In fact, it is easy to see that the quotient manifold T^*N/W, *i.e.*, the manifold of leaves of the foliation of T^*N defined by

W, is simply the vector bundle T^*N/C^0, quotient of the cotangent bundle T^*N by its vector sub-bundle C^0, annihilator of C.

Since the submanifold D has a transverse intersection with each leaf of the foliation of T^*N defined by W, that submanifold can be identified, at least locally, with the manifold of leaves T^*N/W.

Let us now determine the symplectic orthogonal of W. Let $z \in T^*N$, and $v \in T_z(T^*N)$. We have $v \in (\operatorname{orth} W)_z$ if and only if $\langle q^*\eta, v\rangle = 0$ for all $\eta \in C^0_{q(z)}$, that means, if and only if $Tq(v) \in C_{q(z)}$. Therefore,

$$\operatorname{orth} W = (Tq)^{-1}(C).$$

Observe now that C is a completely integrable vector sub-bundle of TN if and only if $(Tq)^{-1}(C) = \operatorname{orth} W$ is a completely integrable vector sub-bundle of $T(T^*N)$. When that is true, Libermann's Theorem 2 shows that there exists on T^*N/W a unique Poisson structure for which the canonical projection $T^*N \to T^*N/W$ is a Poisson map. But we have seen that at least locally, T^*N/W can be identified with the submanifold D, and under that identification, Λ_D is the Poisson tensor of T^*N/W, and therefore, is a true Poisson tensor.

Conversely, let us assume that Λ_D is a Poisson tensor. Here our proof follows closely the original one given by Van der Schaft and Maschke [20], but in a more intrinsic, coordinate free form. Every vector field X on N can be considered as a smooth function on T^*N, whose restriction to each fibre is a linear form; we will denote that function by f_X, in order to distinguish whether X is considered as a vector field or as a function. According to well known properties (see for example [13], chapter III, exercise 17.5), the Hamiltonian vector field $\Lambda^\sharp(df_X)$ on T^*N is projectible by $q : T^*N \to N$, and has X as its projection on N; moreover, for two vector fields X and Y on N, the Poisson bracket $\{f_X, f_Y\}$ is equal to $f_{[X,Y]}$. When the vector field X is a section of C, the corresponding function f_X is constant on each leaf of the foliation defined on T^*Q by W. Therefore, f_X can be considered as a function on T^*N/W. Using the direct sum deconmposition $T_D(T^*N) = TD \oplus W_D$, we can split the Hamiltonian vector field $\Lambda^\sharp df_X$, restricted to D, as

$$(\Lambda^\sharp df_X)|_D = (\Lambda^\sharp df_X)_D + (\Lambda^\sharp df_X)_W,$$

where in the right hand side the first term is a vector field on the submanifold D and the second term a section of W_D. Using the very definition of Λ_D, we see that

$$\Lambda^\sharp_D d\big(f_X|_D\big) = (\Lambda^\sharp df_X)_D.$$

We observe that $q : T^*N \to N$, when restricted to D, is a submersion of D onto an open subset of N. The projections on N of $(\Lambda^\sharp df_X)_D$ and of $(\Lambda^\sharp df_X)|_D$ are equal, since the projection of the other term $(\Lambda^\sharp df_X)_W$ vanishes. Therefore, the projection on N of $\Lambda^\sharp_D d(f_X|_D)$ is X.

Now we use the fact that Λ_D is a Poisson tensor. For two sections X and Y of C, we have

$$\big[\Lambda^\sharp_D d(f_X|_D), \Lambda^\sharp_D d(f_Y|_D)\big] = \Lambda^\sharp_D d\{f_X|_D, f_Y|_D\}_D.$$

Since the projections on N of $\Lambda^\sharp_D d(f_X|_D)$ and $\Lambda^\sharp_D d(f_Y|_D)$ are X and Y, respectively, we

see that the bracket $[X, Y]$ belongs to the projection on N of the span of $\Lambda_D^\sharp$. But it is easy to see that the projection on N of the span of $\Lambda_D^\sharp$ is C. Therefore, for every pair (X, Y) of sections of C, the bracket $[X, Y]$ is a section of C, and we conclude that C is completely integrable.

Remark. The fact that the submanifold D of T^*N is the image of C under the Legendre transformation $\mathcal{L} : TN \to T^*N$ does not play an important role for what regards the properties of the tensor Λ_D: any submanifold of T^*Q whose intersection with every leaf of the foliation defined by W is transverse can be locally identified with the quotient T^*N/W, and therefore has the same properties as D. However, the precise definition of D is important when we compare $\Lambda_D^\sharp d(H|_D)$ with the projection on D of $(\Lambda^\sharp dH)|_D$: these two vector fieds on D would not be equal in general if D were not the image of C by the Legendre transformation. This fact, and other properties of the pseudo-Poisson tensor Λ_D are discussed at length by Cantrijn *et al.* [1].

3. Foliated Poissson manifolds. We will consider in this section foliations of a Poisson manifold whose properties generalize, in a sense, Libermann's Theorem 2. We will need some properties of Poisson manifolds, whose proofs may be found for example in the book by I. Vaisman [19]. Let us recall them briefly, just to indicate the appropriate notations and sign conventions.

Let (P, Λ) be a Poisson manifold, and $\Omega(P) = \oplus_k \Omega^k(P)$ be the exterior algebra of differential forms on P. There exists on $\Omega(P)$ a graded Lie algebra structure, for which the bracket of a k-form η and a l-form ζ is the $k + l - 1$-form $[\eta, \zeta]$ given by

$$[\eta, \zeta] = (-1)^k \big(\Delta(\eta \wedge \zeta) - (\Delta\eta) \wedge \zeta - (-1)^k \eta \wedge (\Delta\zeta)\big).$$

We have set $\Delta = i(\Lambda)d - di(\Lambda)$. By definition $i(\Lambda)$ is the graded endomorphism of degree -2 of $\Omega(P)$ such that, for $\eta \in \Omega^k(P)$ and $Q \in A^{k-2}(P)$ (space of $(k-2)$-multivectors on P),

$$\big\langle i(\Lambda)\eta, Q\big\rangle = \langle \eta, \Lambda \wedge Q\rangle.$$

The space $\Omega^1(P)$ of differential 1-forms on P is stable by that bracket, for which it is a Lie algebra. The bracket of two differential 1-forms α and β is the differential 1-form

$$[\alpha, \beta] = -d\Lambda(\alpha, \beta) + \mathcal{L}(\Lambda^\sharp\alpha)\beta - \mathcal{L}(\Lambda^\sharp\beta)\alpha.$$

When $\alpha = df$ and $\beta = dg$ are exact 1-forms, their bracket is related to the Poisson bracket $\{f, g\}$ by

$$[df, dg] = d\{f, g\}.$$

The cotangent bundle T^*P equipped with the bracket of 1-forms on the space of its sections is a Lie algebroid in the sense of Pradines [18], with anchor map $\Lambda^\sharp : T^*P \to TP$. It means that when extended to sections, $\Lambda^\sharp$ is a Lie algebra homomorphism and that, for α and $\beta \in \Omega^1(P)$ and $f \in C^\infty(P, \mathbf{R})$,

$$[\alpha, f\beta] = \big(\mathcal{L}(\Lambda^\sharp\alpha)f\big)\beta + f[\alpha, \beta].$$

Let $A(P) = \oplus_k A^k(P)$ be the graded exterior algebra of multivectors on P. Let us denote by $[Q, R]$ the Schouten-Nijenhuis bracket of the two elements Q and R of $A(P)$,

and by $d_\Lambda : A(P) \to A(P)$ the graded endomorphism of degree 1:

$$d_\Lambda(Q) = [\Lambda, Q].$$

Then d_Λ is a cohomology operator, *i.e.*, it satisfies $d_\Lambda \circ d_\Lambda = 0$. The corresponding cohomology is called the Lichnerowicz-Poisson cohomology of (P, Λ). When extended naturally to sums of exterior products, the anchor map $\Lambda^\sharp$ becomes an exterior algebras homomorphism from $\Omega(P)$ into $A(P)$. Moreover, $\Lambda^\sharp$ is a graded Lie algebras homomorphism (when $A(P)$ is equipped with the graded Lie algebra structure defined by the Schouten-Nijenhuis bracket) and a differntial complex homomorphism from $\big(\Omega(P), d\big)$ into $\big(A(P), d_\Lambda\big)$. It means that, for η and $\zeta \in \Omega(P)$,

$$\Lambda^\sharp\big([\eta, \zeta]\big) = \big[\Lambda^\sharp\eta, \Lambda^\sharp\zeta\big], \quad \Lambda^\sharp(d\eta) = d_\Lambda(\Lambda^\sharp\eta) = [\Lambda, \Lambda^\sharp\eta].$$

Remarks. The Lie algebra structure of $\Omega^1(P)$ was discovered by Gel'fand and Dorfman [6] and independently Magri and Morosi [16]. It was extended by Koszul [11] into a graded Lie algebra structure on $\Omega(P) = \oplus_{k\in\mathbf{N}}\Omega^k(P)$. The concept of a Lie algebroid is due to Pradines [18], and the Lie algebroid structure of T^*P was obtained by Coste, Dazord, Weinstein and Sondaz [3, 5]. The Lichnerowicz-Poisson cohomology was discovered by A. Lichnerowicz [14].

We may now state the following result.

Proposition 1. *Let (P, Λ) be a Poisson manifold, Y a vector sub-bundle of TP, Y^0 its annihilator and $W = \Lambda^\sharp(Y^0)$. Then W is a smooth generalized distribution on P (which may not be of constant rank), and we have the following properties:*

(i) *if for every pair (α, β) of smooth sections of Y^0, the bracket $[\alpha, \beta]$ (for the Lie algebra structure of $\Omega^1(P)$) is a section of Y^0, then W is involutive;*

(ii) *if for every smooth section α of Y^0, the exterior differential $d\alpha$ belongs to the ideal generated by the space of smooth sections of Y^0 or, equivalently, if Y is a completely integrable vector sub-bundle of TP, then for every smooth section X of W, the Lie derivative $\mathcal{L}(X)\Lambda$ belongs to the ideal generated by the space of smooth sections of W.*

When Y^0 satisfies Property (i) *and when, in addition, W is of constant rank, the vector sub-bundle W is completely integrable. When in addition Y^0 satisfies Property* (ii) *and when the space P/W of leaves of the foliation of P defined by W has a smooth manifold structure for which the canonical projection $\pi : P \to P/W$ is a submersion, there exists on P/W a unique Poisson structure for which π is a Poisson map.*

Proof. The generalized distribution W is smooth, since it is spanned by the smooth vector fields $\Lambda^\sharp\alpha$, for all smooth sections α of Y^0.

Let us assume that Y^0 satisfies Property (i). For a pair (α, β) of smooth sections of Y^0, we have

$$\Lambda^\sharp[\alpha, \beta] = [\Lambda^\sharp\alpha, \Lambda^\sharp\beta].$$

Since $[\alpha, \beta]$ is a section of Y^0, $[\Lambda^\sharp\alpha, \Lambda^\sharp\beta]$ is a section of $\Lambda^\sharp Y^0 = W$. Since W is generated by the smooth vector fields $\Lambda^\sharp\gamma$, for all smooth sections γ of Y^0, we see that W is

involutive (*i.e.*, the set of its smooth sections is closed under the bracket operation). If in addition W is of constant rank, by Frobenius' theorem it is completely integrable.

Let us assume that Y^0 satisfies Property (ii): for every smooth section α of Y^0, $d\alpha$ belongs to the ideal generated by the space of smooth sections of α. By the covariant version of Frobenius' theorem, that property is equivalent to the complete integrability of Y. Le X be a smooth section of $W = \Lambda^\sharp(Y^0)$. There exists (at least locally, in a neighbourhood of each point of P) a smooth section α of Y^0 such that $X = \Lambda^\sharp\alpha$. Using the properties of Poisson manifolds indicated above, we have

$$\mathcal{L}(X)\Lambda = [X, \Lambda] = -[\Lambda, X] = -d_\Lambda X = -\Lambda^\sharp(d\alpha).$$

But $d\alpha$ belongs to the ideal generated by the space of smooth sections of Y^0, and the above formula proves that $\mathcal{L}(X)\Lambda$ belongs to the ideal generated by the set of smooth sections of $\Lambda^\sharp(Y^0)$, that means, by the set of smooth sections of W.

When Y^0 satisfies both Properties (i) and (ii), and when in addition P/W has a smooth manifold stucture for which the canonical projection is a submersion, Property (ii) shows that the tensor field Λ can be projected onto P/W. Let $\Lambda_{P/W}$ be its projection. By well known properties of the Schouten-Nijenhuis bracket, $[\Lambda_{P/W}, \Lambda_{P/W}]$ is the projection of $[\Lambda, \Lambda]$. Since Λ is a Poisson tensor, $[\Lambda, \Lambda] = 0$, and therefore $[\Lambda_{P/W}, \Lambda_{P/W}] = 0$, which proves that $\Lambda_{P/W}$ is a Poisson tensor. By the very definition of $\Lambda_{P/W}$, the projection $\pi : P \to P/W$ is a Poisson map. ∎

Remark. Under the assumptions of the above Proposition, one can easily prove that the characteristic distribution of P/W (*i.e.*, the span $\Lambda^\sharp_{P/W}\big(T^*(P/W)\big)$ of $\Lambda^\sharp_{P/W}$) is the image, by the projection $\pi : P \to P/W$, of $Y \cap \Lambda^\sharp(T^*P)$. Observe that it is a generalized distibution, which may not be of constant rank.

4. On submanifolds of a Poisson manifold. We consider in this section a submanifold D of a Poisson manifold (P, Λ) and a vector sub-bundle W of T_DP complementary to TD. We therefore have the direct sum decomposition

$$T_DP = TD \oplus W.$$

As indicated in the Introduction, we define a two-times contravariant, skew-symmetric tensor field Λ_D on D by setting, for each $z \in D$ and each η and $\zeta \in T^*_zD$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\widehat{\eta}, \widehat{\zeta}),$$

where $\widehat{\eta}$ is defined by

$$\langle \widehat{\eta}, v\rangle = \begin{cases} \langle \eta, v\rangle & \text{if } v \in T_zD, \\ 0 & \text{if } v \in W_z, \end{cases}$$

and where $\widehat{\zeta}$ is defined by a similar formula.

By using a suitable tubular neighbourhood of D in P, we see that there exists a submersion π of an open neighbourhood U of D in P, onto D, such that $\pi|_D = \mathrm{id}_D$ and, for all $z \in D$,

$$T_z\pi(v) = \begin{cases} v & \text{if } v \in T_zD, \\ 0 & \text{if } v \in W_z. \end{cases}$$

Of course, (U, π) is not unique.

The following lemma indicates a necessary and sufficient condition under which Λ_D is a Poisson tensor.

LEMMA 1. *Let $i_D : D \to P$ be the canonical injection. For every pair (f, g) of smooth functions on D, let $A(f, g)$ be the smooth function, defined on the open neighbourhood U of D in P, by*

$$A(f, g) = \pi^* i_D^* \{\pi^* f, \pi^* g\} - \{\pi^* f, \pi^* g\}.$$

The bilinear map A takes its values in the ideal of smooth functions on U which vanish on D, and the tensor field Λ_D is Poisson if and only if, for every triple (f, g, h) of smooth functions on D,

$$i_D^*(\{A(f, g), \pi^* h\} + \{A(g, h), \pi^* f\} + \{A(h, f), \pi^* g\}) = 0.$$

PROOF. Let us first observe that for any pair (f, g) of smooth functions on D, the functions $\pi^* i_D^* \{\pi^* f, \pi^* g\}$ and $\{\pi^* f, \pi^* g\}$ are equal on D; therefore $A(f, g)$ vanishes on D.

The bracket $\{f, g\}_D$ of two smooth functions f and g on D being defined by $\{f, g\}_D = \Lambda_D(df, dg)$ we have, for every pair $(f, g,)$ of smooth function on D,

$$\{f, g\}_D = i_D^* \{\pi^* f, \pi^* g\},$$

and therefore, for every triple (f, g, h) of smooth functions on D,

$$\{\{f, g\}_D, h\}_D = i_D^* \{\pi^* i_D^* \{\pi^* f, \pi^* g\}, \pi^* h\} = i_D^* \{A(f, g) + \{\pi^* f, \pi^* g\}, \pi^* h\}.$$

Since the bracket of functions on U satisfies the Jacobi identity, we get after summation

$$\begin{aligned} &\{\{f, g\}_D, h\}_D + \{\{g, h\}_D, f\}_D + \{\{h, f\}_D, g\}_D \\ &\qquad = i_D^*(\{A(f, g), \pi^* h\} + \{A(g, h), \pi^* f\} + \{A(h, f), \pi^* g\}). \end{aligned}$$

The result follows immediately. ∎

The following Lemma indicates another form of the same necessary and sufficient condition, involving the bracket of 1-forms on P.

LEMMA 2. *For every pair (α, β) of closed 1-forms on D, let $B(\alpha, \beta)$ be the smooth vector field on D:*

$$B(\alpha, \beta) = T\pi \circ \Lambda^\sharp (\pi^* i_D^* - \mathrm{id})[\pi^* \alpha, \pi^* \beta] \circ i_D.$$

The map B is bilinear, skew-symmetric and the tensor field Λ_D is Poisson if and only if, for every triple (α, β, γ) of closed 1-forms on D,

$$\big\langle \gamma, B(\alpha, \beta) \big\rangle + \big\langle \alpha, B(\beta, \gamma) \big\rangle + \big\langle \beta, B(\gamma, \alpha) \big\rangle = 0.$$

PROOF. Let f, g and h be three smooth functions on D. We have

$$\begin{aligned} i_D^* \big\{A(f, g), \pi^* h\big\} &= \big\langle d(\pi^* h), \Lambda^\sharp d\, A(f, g) \big\rangle \circ i_D \\ &= \big\langle dh, T\pi \circ \Lambda^\sharp (\pi^* i_D^* - \mathrm{id}) d\{\pi^* f, \pi^* g\} \big\rangle \circ i_D \\ &= \big\langle dh, T\pi \circ \Lambda^\sharp (\pi^* i_D^* - \mathrm{id})[\pi^* df, \pi^* dg] \big\rangle \circ i_D \\ &= \big\langle dh, B(df, dg) \big\rangle. \end{aligned}$$

Closed 1-forms on D are locally exact, and the properties involved are local. Therefore, Lemma 2 follows directly from Lemma 1. ∎

Remarks. The necessary and sufficient condition indicated in Lemma 2 is clearly a cohomology condition expressing that the map B is a cocycle with values in the vector space of smooth vector fields on D. Unfortunately, the definition of B involves some amount of arbitrariness (the choice of the submersion π). It would be nice to obtain a condition without any arbitrary choice.

The conditions indicated in Lemmas 1 and 2 are necessary and sufficient, but not very easy to handle. However, the following Proposition indicates a condition much easier to handle, but only sufficient.

Proposition 2. *If for every pair (α,β) of closed 1-forms on D, we have $B(\alpha,\beta)=0$, then Λ_D is a Poisson tensor field.*

Proof. It is an immediate consequence of Lemma 2. ∎

Application. As we shall see, the Poisson submanifolds of the first and second kind found by A. Weinstein, as well as the Poisson submanifolds of Van der Schaft and Maschke, are examples of applications of Proposition 2.

Poisson submanifolds of the first kind. Let D be a Poisson submanifold of the first kind of (P,Λ). Then for every pair (α,β) of 1-forms on D, $(\pi^* i_D^* - \mathrm{id})[\pi^*\alpha,\pi^*\beta]\circ i_D$ is a section of $(TD)^0$. But TD contains the span of $\Lambda^\sharp$ along D, therefore $(TD)^0$ is contained in the kernel of $\Lambda^\sharp$ along D. For that reason, $\Lambda^\sharp(\pi^* i_D^* - \mathrm{id})[\pi^*\alpha,\pi^*\beta]\circ i_D = 0$, which implies that $B(\alpha,\beta)=0$.

Poisson submanifolds of the second kind. Let D be a Poisson submanifold of the second kind of (P,Λ). Then $W=\Lambda^\sharp\big((TD)^0\big)$, and for every pair (α,β) of 1-forms on D, $\Lambda^\sharp\circ(\pi^* i_D^* - \mathrm{id})[\pi^*\alpha,\pi^*\beta]\circ i_D$ is a section of W, which is the kernel of $T_D\pi$. Therefore $T\pi\circ\Lambda^\sharp(\pi^* i_D^* - \mathrm{id})[\pi^*\alpha,\pi^*\beta]\circ i_D = 0$, that means $B(\alpha,\beta)=0$.

Poisson submanifolds of Van der Schaft and Maschke. Let D be a Poisson submanifold of the cotangent bundle T^*N of the type considered by Van der Schaft and Maschke. Let us take for π the canonical projection of T^*Q onto the quotient manifold T^*Q/W, identified with D. Then for every pair (α,β) of 1-forms on D, $(\pi^* i_D^* - \mathrm{id})[\pi^*\alpha,\pi^*\beta]=0$. Therefore $B(\alpha,\beta)=0$.

5. On submanifolds of a Jacobi manifold. Let us first recall that a Jacobi manifold is a smooth manifold M whose space of smooth functions $C^\infty(M,\mathbf{R})$ is equipped with a Lie algebra structure, the bracket $\{f,g\}$ of two smooth functions f and g being given by a local bilinear operator (it means that if f, or g, vanishes on some open subset of M, then $\{f,g\}$ vanishes on that subset). A. Kirillov [9] has shown that on a Jacobi manifold M, there exists a smooth vector field E and a smooth two times contravariant skew-symmetric tensor Λ such that, for every pair (f,g) of smooth functions on M,

$$\{f,g\} = \Lambda(df,dg) + \langle f\,dg - g\,df, E\rangle. \tag{$*$}$$

A. Lichnerowicz [15] has shown that E and Λ must satisfy

$$[E,\Lambda]=0, \qquad [\Lambda,\Lambda]=2E\wedge\Lambda, \tag{$**$}$$

the bracket in these expressions being the Schouten bracket. A. Lichnerowicz has shown that conversely, if on a manifold M a smooth vector field E and a smooth two times contravariant skew-symmetric tensor Λ satisfy $(**)$, the bracket of functions given by $(*)$ satifies the Jacobi identity, and therefore defines on M a Jacobi manifold structure. The manifold M with that structure will be denoted by (M, Λ, E).

Let (M, Λ, E) be a Jacobi manifold. We define on $P = \mathbf{R} \times M$ a two times contravariant skew-symmetric tensor Λ_P by setting

$$\Lambda_P = t\left(\Lambda - t\,\frac{\partial}{\partial t} \wedge E\right)$$

where t is the canonical coordinate on the factor $\mathbf{R}$ of $\mathbf{R} \times E$. A. Lichnerowicz [15] has shown that Λ_P is a Poisson tensor on P, which satisfies

$$[Z, \Lambda_P] = -\Lambda_P, \qquad \text{with} \qquad Z = -t\,\frac{\partial}{\partial t}.$$

Let $J^1(M, \mathbf{R})$ be the bundle of 1-jets of smooth functions on M. A smooth section of that bundle is a pair (σ, η), where $\sigma \in C^\infty(M, \mathbf{R})$ is a smooth function and $\eta \in \Omega^1(M)$ is a smooth 1-form on M. The map

$$(\sigma, \eta) \mapsto \omega = \frac{1}{t}\,\eta - \frac{\sigma}{t^2}\,dt$$

associates, with every smooth section (σ, η) of $J^1(M, \mathbf{R})$, a smooth 1-form on the open subset $(\mathbf{R}\backslash\{0\}) \times M$ of the Poisson manifold $P = \mathbf{R} \times M$. That map is injective and its image is the set of smooth 1-forms ω on $(\mathbf{R}\backslash\{0\}) \times M$ which satisfy $\mathcal{L}(Z)\omega = \omega$. That image is a vector subspace of the set of smooth 1-forms which is invariant by the bracket of 1-forms. Using that property, it is easy to prove the following theorem, due to Y. Kerbrat and Z. Souici-Benhammadi [8]:

THEOREM 3 (Kerbrat and Souici-Benhammadi [8]). *Let (M, Λ, E) be a Jacobi manifold. The bundle $J^1(M, \mathbf{R})$ of 1-jets of smooth functions on M has a Lie algebroid structure, with anchor map*

$$(\sigma, \eta) \mapsto \Lambda^\sharp \eta + \sigma E,$$

the bracket $(\sigma', \eta') = \big[(\sigma_1, \eta_1), (\sigma_2, \eta_2)\big]$ of two sections of $J^1(M, \mathbf{R})$ being given by the formulae:

$$\begin{aligned}
\sigma' &= -\Lambda(\eta_1, \eta_2) + i(\Lambda^\sharp \eta_1 + \sigma_1 E)d\sigma_2 - i(\Lambda^\sharp \eta_2 + \sigma_2 E)d\sigma_1,\\
\eta' &= \mathcal{L}(\Lambda^\sharp \eta_1 + \sigma_1 E)\eta_2 - \mathcal{L}(\Lambda^\sharp \eta_2 + \sigma_2 E)\eta_1\\
&\qquad - \langle \eta_1, E\rangle(\eta_2 - d\sigma_2) + \langle \eta_2, E\rangle(\eta_1 - d\sigma_1) - d\big(\Lambda(\eta_1, \eta_2)\big).
\end{aligned}$$

By using the Poisson manifold $P = \mathbf{R} \times M$ associated with the Jacobi manifold (M, Λ, E), it should be possible to obtain, for foliations and for submanifolds of a Jacobi manifold, results similar to those obtained in Sections 3 and 4 for foliations and submanifolds of a Poisson manifold. Let us recall in particular the result, already obtained in [4], which generalizes Poisson submanifolds of the second kind:

PROPOSITION 3. *Let D be a submanifold of the Jacobi manifold (M, Λ, E). We assume that, for each $x \in D$,*

$$T_x M = T_x D \oplus \Lambda^\sharp(TxD^0).$$

For every 1*-form* η *on* D*, we denote by* $\widehat{\eta}$ *the section of* $T^*_D M$ *defined by*

$$\langle \widehat{\eta}, v\rangle = \begin{cases} \langle \eta, v\rangle & \text{if } v \in T_x D, \\ 0 & \text{if } v \in \Lambda^\sharp(T_x D^0), \end{cases} \qquad \text{with} \quad x \in D.$$

We define a vector field E_D *and a two times contravariant skew-symmetric tensor* Λ_D *on* D *by setting, for all* 1*-forms* η *and* ζ *on* D*,*

$$\langle \eta, E_D\rangle = \langle \widehat{\eta}, E\rangle, \qquad \Lambda_D(\eta, \zeta) = \Lambda(\widehat{\eta}, \widehat{\zeta}).$$

Then (D, Λ_D, E_D) *is a Jacobi manifold.*

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