Causal nets for geometrical Gandy–Păun–Rozenberg machines

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Abstract. An approach to the computational complexity beyond the known complexity measures of the consumed time and space of computation is proposed. The approach focuses on the chaotic behavior and randomness aspects of computational processes and is based on a representation of these processes by causal nets.

1 Introduction

A certain new approach to the investigations of the computational complexity of abstract systems allowing some unrestricted parallelism of computation is proposed, where the computational processes realized in a discrete time with a central clock by these systems are represented by causal nets similar to those in [4] and related to causal sets in [1].

The representation of computational processes by causal nets is aimed to provide an abstraction from those features of computational processes which do not have a spatial nature such that the abstraction could make visible some new aspects of the processes like an aspect of chaotic behavior or a fractal shape.

The aspects of a chaotic behavior and a fractal shape inspired by the research area of dynamics of nonlinear systems [20] regarding an unpredictability of the behavior of these systems[1] could suggest an answer to the following question formulated in [21]: *Is the concept of randomness, founded in the concept of absence of computable regularities, the only adequate and consistent one? In which direction, if any, should one look for alternatives?*

The answers may have an impact on designing pseudorandom number generators, cf. [23], [24], applied in statistics, cryptography, and Monte Carlo Method.

The proposed approach is aimed to provide a possibly uniform precise mathematical treatment of causal nets and related concepts which could serve for measuring of complexity of computational processes by a use of graph dimensions [13] and network fractal dimensions [19], [7], [8] in parallel to measuring complexity of random strings in [11] by Hausdorff dimension.

The proposed approach concerns the investigations of abstract computing devices which are geometrical Gandy–Păun–Rozenberg machines.

[1] unpredictability due to sensitive dependence on initial conditions—an important feature of deterministic transient chaos [20] often having fractal shape.

The geometrical Gandy–Păun–Rozenberg machines are some modifications of the known Gandy–Păun–Rozenberg machines [14], [15].

The assumption that the sets of instantaneous descriptions of geometrical Gandy–Păun–Rozenberg machines are skeletal sets of finite directed graphs together with the features of machine local causation rewriting rules provide a natural construction of causal nets representing computational processes.

2 Geometrical Gandy–Păun–Rozenberg machines

We refer the reader to Appendix A and Appendix B (quoting the main definitions of [14], [15]) for unexplained notions and notation concerning labelled directed graphs, Gandy–Păun–Rozenberg machines (briefly G–P–R machines), and generalized G–P–R machines.

We recall the main difference between G–P–R machines and generalized G–P–R machines:

– the auxiliary rules of the G–P–R machines are not specified and for every G–P–R machine $\mathcal{M}$ with its transition function $\mathcal{F}_{\mathcal{M}}$ and for every instantaneous description G of $\mathcal{M}$ the instantaneous description $\mathcal{F}_{\mathcal{M}}(G)$ is a colimit of the gluing diagram $\mathcal{D}^G$ determined by the set $\mathcal{P}\ell(G)$ of maximal applications of the machine rewriting rules to G.
– the generalized G–P–R machines are equipped with auxiliary rules besides the rewriting rules and for every generalized G–P–R machine $\mathcal{M}$ with its transition function $\mathcal{F}_{\mathcal{M}}$ and for every instantaneous description G of $\mathcal{M}$ the instantaneous description $\mathcal{F}_{\mathcal{M}}(G)$ is a colimit of the generalized gluing diagram $\mathcal{D}_G$ determined by both the machine rewriting rules and the auxiliary rules.

Definition 2.1. We define a *simple geometrical G–P–R machine* and a *strict geometrical G–P–R machine* to be the modifications of a G–P–R machine and a generalized G–P–R machine, respectively, such that

– in both cases of a simple and a strict machine we assume that
  - the set of labels of vertices of the directed graphs belonging to the set of instantaneous descriptions of a given machine is a one element set or equivalently these graphs are not labelled at all, an analogous assumption concerns the graphs appearing in the machine rewriting rules and auxiliary rules,
  - for every machine $\mathcal{M}$ there exists a natural number $n > 0$ such that for every graph G belonging to the set of instantaneous descriptions of $\mathcal{M}$ the set $V(G)$ of vertices of G is a set of ordered n-tuples of elements of $Q^\bullet$, where $Q^\bullet$ is the set of rational numbers, if necessary, extended to the recursive real numbers which are linear combinations of $\sqrt{2}$, $\sqrt{3}$, etc. with rational coefficients (hence we use the adjective 'geometrical'),

– in the case of a simple machine we impose a strenghtening that the graphs belonging to the set of instantaneous descriptions of the machine or appearing in the conclusions of the machine rewriting rules are not necessarily isomorphically perfect graphs.

Theorem. *For both cases of a simple and of a strict geometrical G–P–R machine if the set of its instantaneous description is a recursive set, then the transition function of the machine is a computable function.*

Proof. We prove the theorem for the case of a simple geometrical G–P–R machine $\mathcal{M}$ with its transition function $\mathcal{F}_{\mathcal{M}}$.

The following assignments are computable:

– the assignment to a finite gluing diagram its colimit constructed as in Appendix A in the domain of finite directed graphs,
– the assignment to an instantaneous description G of $\mathcal{M}$ the set $\mathcal{P}\ell(G)$ of maximal applications of the rewriting rules of $\mathcal{M}$,
– the assignment to an instantaneous description G of $\mathcal{M}$ the gluing diagram $\mathcal{D}^G$ which is determined by $\mathcal{P}\ell(G)$ in an effective way.

Hence the assignment to an instantaneous description G of $\mathcal{M}$ the result of the construction of a colimit of the gluing diagram $\mathcal{D}^G$ is also computable assignment. Therefore an effective search of that unique instantaneous description $G' = \mathcal{F}_{\mathcal{M}}(G)$ which is isomorphic to the above result of the construction of a colimit of the gluing diagram $\mathcal{D}^G$ suffices for reaching $\mathcal{F}_{\mathcal{M}}(G)$ in an effective way. This effective search is provided by the assumption that the set of instantaneous descriptions of the machine $\mathcal{M}$ is a recursive set. Thus the transition function $\mathcal{F}_{\mathcal{M}}$ is a computable function.

The proof of the theorem for the case of strict geometrical G–P–R machines is similar to the above proof.

Examples 2.1 (The simulation of cellular automata). The generalized G–P–R machine $\mathcal{M}^{\mathrm{SGL}}$ in [15] is an example of a strict geometrical G–P–R machine, where $\mathcal{M}^{\mathrm{SGL}}$ simulates the spatial and temporal behavior of a cellular automaton identified with the eastern expansion fragment of Conway's *Game of life*.

We show now an example of a simple geometrical G–P–R machine which is aimed to simulate the behavior of one-dimensional cellular automaton with two cell states $0, 1$ and with the rule 30 given by the formula

$$a_{i-1} \,\mathrm{xor}\, (a_i \,\mathrm{or}\, a_{i+1}),$$

cf. [23], [24], where xor is 'exclusive or'. This simple geometrical G–P–R machine, denoted by $\mathcal{M}^{30}$, is defined in the following way.

The instantaneous descriptions and the rewriting rules of $\mathcal{M}^{30}$ are defined by using the finite directed graphs cl^n_x (for an integer n and $x \in \{0, 1, !, \varnothing\}$) corresponding to the single cells for $x \in \{0, 1, !\}$, where cl^n_x are such that:

– the graph $\mathrm{cl}^n_\varnothing$ is the square

$$\begin{array}{ccc} (n,1) & \longrightarrow & (n+1,1) \\ \uparrow & & \uparrow \\ (n,0) & \longrightarrow & (n+1,0) \end{array}$$

together with the loop $\big((0,0),(0,0)\big)$ in the case $n=0$, and with the path from $(n+1,0)$ to $(n,1)$ containing three intermediate vertices $(n+1-\frac{i}{4},\frac{i}{4})$ with $\{1,2,3\}$,

– the graph cl^n_x for $x \in \{0,1,!\}$ consists of:
  - the graph $\mathrm{cl}^n_\varnothing$ as a subgraph,
  - the edge $\big((n,0),(n+1-\frac{x+1}{4},\frac{x+1}{4})\big)$ for $x \in \{0,1\}$, indicating that the graph cl^n_x corresponds to a cell in state x and called an *edge indicating a state of a cell*,
  - the edge $\big((n,0),(n+1-\frac{3}{4},\frac{3}{4})\big)$ for $x =!$.

An instantaneous description of $\mathcal{M}^{30}$ is that graph G which is the graph union (cf. Appendix A)

$$\mathrm{cl}^i_! \cup \Big(\bigcup_{i<k<j} \mathrm{cl}^k_{x_k}\Big) \cup \mathrm{cl}^j_!$$

for some integers $i < -1$, $j > 1$ and for some family $\mathrm{cl}^k_{x_k}$ $(i < k < j)$ with $x_k \in \{0,1\}$ for all integers k such that $i < k < j$.

The rewriting rules of $\mathcal{M}^{30}$ are given by

– $\mathrm{cl}^1_i \cup \mathrm{cl}^2_j \cup \mathrm{cl}^3_k \vdash \mathrm{cl}^1_\varnothing \cup \mathrm{cl}^2_{\rho(i,j,k)} \cup \mathrm{cl}^3_\varnothing$ for $\{i,j,k\} \subseteq \{0,1\}$, where $\rho(i,j,k) = i \,\mathrm{xor}\, (j \,\mathrm{or}\, k)$,
– $\mathrm{cl}^2_! \cup \mathrm{cl}^3_j \cup \mathrm{cl}^4_k \vdash \mathrm{cl}^1_! \cup \mathrm{cl}^2_{\rho(0,0,j)} \cup \mathrm{cl}^3_{\rho(0,j,k)} \cup \mathrm{cl}^4_\varnothing$ for $\{j,k\} \subseteq \{0,1\}$,
– $\mathrm{cl}^1_i \cup \mathrm{cl}^2_j \cup \mathrm{cl}^3_! \vdash \mathrm{cl}^1_\varnothing \cup \mathrm{cl}^2_{\rho(i,j,0)} \cup \mathrm{cl}^3_{\rho(j,0,0)} \cup \mathrm{cl}^4_!$ for $\{i,j\} \subseteq \{0,1\}$,
– the identity rule $\circlearrowleft\bullet\, 0 \vdash \circlearrowleft\bullet\, 0$, where $\circlearrowleft\bullet\, 0$ is the graph with single vertex 0 and with single edge which is a loop.

The graphs cl^2_j and cl^3_j appearing in the middle of the premises of the above rules are called the *centers* of these rules, respectively.

The one-dimensional cellular automata in [23], [24] and small Turing machines in [7], [8] can be simulated by simple G–P–R machines constructed in a similar way to the machine $\mathcal{M}^{30}$.

Example 2.2 (generation of the contours of the iterations of fractals). We show a simple geometrical G–P–R machine whose single rewriting rule serves for generating the contours of the iterations of Sierpiński gasket. This machine, denoted by $\mathcal{M}^{\mathrm{Sierp}}$ is defined in the following way.

Let Δ be a directed graph given by

$$V(\Delta) = \big\{(0,0),(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}),(1,0)\big\},$$
$$E(\Delta) = \big\{((0,0),(1,0)),((0,0),(\tfrac{1}{2},\tfrac{\sqrt{3}}{2})),((1,0),(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}))\big\}.$$

The graph Δ is a contour of an equilateral triangle.

The instantaneous descriptions of $\mathcal{M}^{\mathrm{Sierp}}$ are graphs Δ_n (for a natural number $n \geq 0$) defined inductively by

$$\Delta_0 = \big(V(\Delta), E(\Delta) \cup \{((0,1),(0,1))\}\big),$$
$$\Delta_{n+1} = \bigcup_{i\in\{1,2,3\}} f_i(\Delta_n),$$

where f_1, f_2, f_3 are functions forming the iterated function system for Sierpiński gasket, cf. [18] and Appendix C, and for a directed graph G with $V(G) \subset Q^{\bullet}\times Q^{\bullet}$ $f_i(G)$ is a graph such that

$$V(f_i(G)) = \big\{f_i(v) \,|\, v \in V(G)\big\},$$
$$E(f_i(G)) = \big\{(f_i(v), f_i(v')) \,|\, (v,v') \in E(G)\big\}.$$

A unique rewriting rule of $\mathcal{M}^{\mathrm{Sierp}}$ is of the form

$$\Delta_1 \vdash \Delta_2.$$

The graphs Δ_n $(n > 0)$ are the contours of the iterations of Sierpiński gasket.

The similar simple G–P–R machines can be constructed for some other fractals determined by iteration function system, e.g. 3D Sierpiński gasket.

3 Causal nets of geometrical G–P–R machines

We propose some precise mathematical treatment of those concepts which express or explicate certain aspects and features of the computational processes realized by geometrical G–P–R machines and which can be investigated within ‘experimental mathematics’ by an analysis (sometimes heuristic) of the plots illustrating those concepts. The plots could be generated by computers like in [24].

Definitions 3.1. For both cases of a simple or of a strict geometrical G–P–R machine $\mathcal{M}$ and an initial instantaneous description G of $\mathcal{M}$ we define an *event with respect to* G to be an ordered pair (v,i) with $v \in V(\mathcal{F}^i_{\mathcal{M}}(G))$ for a natural number $i \geq 0$, where $\mathcal{F}^0_{\mathcal{M}}(G) = G$. Then we define a *full causal relation* $\prec_G$ *with respect to* G to be a binary relation defined on the set $Ev(G)$ of events with respect to G given by

$$(v,i) \prec_G (v',i') \quad \text{iff} \quad i' = i+1 \text{ and there exists } h \in \mathcal{P}\ell(\mathcal{F}^i_{\mathcal{M}}(G))$$

such that $v \in V(\mathrm{im}(h))$ and $v' \in V(\mathrm{im}(q_h))$ for the h-th canonical injection $q_h : \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h)) \to \mathcal{F}^{i+1}_{\mathcal{M}}(G)$ into the colimit of the gluing diagram $\mathcal{D}^{\mathcal{F}^i_{\mathcal{M}}(G)}$ in the simple case and of the generalized gluing diagram $\mathcal{D}_{\mathcal{F}^i_{\mathcal{M}}(G)}$ in the strict case, where $\mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ is the conclusion of the rewriting rule with the premise $\mathrm{dom}(h)$. Thus the ordered pair $\mathcal{N}_G = (Ev(G), \prec_G)$ is called a *full causal net of* $\mathcal{M}$ *with respect to* G.

The proper subnets of the full causal net $\mathcal{N}_G$ with respect to G correspond to various aspects and features of the computation of $\mathcal{M}$ starting with G.

For instance, in the case of Example 2.2 it is worth to consider a *causal net* $\mathcal{N}_G^{\mathrm{gr}}$ *of growth* with *causal growth relation* $\prec_G^{\mathrm{gr}}$ given by

$$\begin{aligned}(v,i) \prec_G^{\mathrm{gr}} (v',i') \quad &\text{iff} \quad i' = i+1 \text{ with } (v,i) \prec_G (v',i') \text{ and both } v \text{ and } v' \\ &\text{are new in } \mathcal{F}^i_{\mathcal{M}}(G) \text{ and in } \mathcal{F}^{i+1}_{\mathcal{M}}(G), \text{ respectively, whenever } i > 0, \\ &\qquad \text{otherwise } v' \text{ is new in } \mathcal{F}^{i+1}(G),\end{aligned}$$

where a vertex x *is new in* $\mathcal{F}^k_{\mathcal{M}}(G)$ if $x \in V(\mathcal{F}^k_{\mathcal{M}}(G))$ and $x \notin V(\mathcal{F}^{k-1}_{\mathcal{M}}(G))$ for $k > 0$.

In the case of the machine in Example 2.2 the projection of $\mathcal{N}^{\mathrm{gr}}_{\Delta_0}$ into the phase space $Q^\bullet \times Q^\bullet$ yields Sierpiński gasket which is a fractal.

In the case of Examples 2.1 it is worth to consider a *causal net* $\mathcal{N}_G^{\mathrm{act}}$ *of activity* with *causal relation* $\prec_G^{\mathrm{act}}$ *of activity* given by

$$\begin{aligned}(v,i) \prec_G^{\mathrm{act}} (v',i') \quad &\text{iff} \quad i' = i+1 \text{ with } (v,i) \prec_G (v',i') \text{ and both } v \text{ and } v' \\ &\text{are the targets of the edges indicating the states} \\ &\text{of the cells in } \mathcal{F}^i_{\mathcal{M}}(G) \text{ and in } \mathcal{F}^{i+1}_{\mathcal{M}}(G), \text{ respectively.}\end{aligned}$$

For the geometrical G–P–R machines simulating one-dimensional cellular automata like the machine $\mathcal{M}^{30}$ in Examples 2.1 one defines the *causal net* $\mathcal{N}_G^{\mathrm{stc}}$ *of strict changes* with the *causal relation* $\prec_G^{\mathrm{stc}}$ of *strict changes* given by

$$\begin{aligned}(v,i) \prec_G^{\mathrm{stc}} (v',i') \quad &\text{iff } i' = i+1 \text{ and there exists } h \in \mathcal{P}\ell(\mathcal{F}^i_{\mathcal{M}}(G)) \\ &\text{such that } h(v_1) = v \text{ and } q_h(v_2) = v' \text{ for those } v_1, v_2 \text{ which are such that} \\ &v_1 \text{ is a vertex of the center of the rule } \mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h)) \\ &\text{and both } v_1, v_2 \text{ are the targets of the edges indicating the state} \\ &\text{of a cell in the premise and in the conclusion of the rule, respectively.}\end{aligned}$$

Thus $\mathcal{N}_G^{\mathrm{stc}}$ is a subnet of $\mathcal{N}_G^{\mathrm{act}}$, moreover, in the case of $\mathcal{M}^{30}$ the plots for the one-dimensional cellular automaton with the rule 30 in [23], [24] illustrate appropriate nets $\mathcal{N}_G^{\mathrm{stc}}$.

The nets $\mathcal{N}_G^{\mathrm{stc}}$ ($G \in \mathcal{S}_{\mathcal{M}}$) coincide with space-time diagrams in [7], [8], where these diagrams are subject of the investigations of computational complexity by using fractal dimension.

The transitive closures $\prec_G^*$, $(\prec_G^x)^*$ ($x \in \{\mathrm{gr}, \mathrm{act}, \mathrm{stc}\}$) give rise to causal sets $\mathcal{C}_G = (Ev(G), \prec_G^*)$ and $\mathcal{C}_G^x = (Ev(G), (\prec_G^x)^*)$ ($x \in \{\mathrm{gr}, \mathrm{act}, \mathrm{stc}\}$) whose logical aspects can be approached like in physics [12] or like in concurrency theory [2].

The investigations of machine $\mathcal{M}^{\mathrm{Sierp}}$ defined in Example 2.2 suggest another approach to the idea of a causal net of a computation of a geometrical G–P–R machine which is introduced in the following definitions.

Definitions 3.2. For both cases of a simple or of a strict geometrical G–P–R machine $\mathcal{M}$ and an initial instantaneous description G of $\mathcal{M}$ we define a *rule application event with respect to* G to be an ordered pair (h, i) with $h \in \mathcal{P}\ell(\mathcal{F}^i_{\mathcal{M}}(G))$ for a natural number $i \geq 0$, where $\mathcal{F}^0_{\mathcal{M}}(G) = G$. Then we define a *rule application causal relation* $\succ^{\mathrm{app}}_G$ *with respect to* G to be a binary relation defined on the set $Ev^{\mathrm{app}}(G)$ of the rule application events with respect to G given by

$$(h, i) \succ^{\mathrm{app}}_G (h', i') \quad \text{iff} \quad i' = i + 1 \text{ and } \operatorname{im}(h) \text{ is a subgraph of } \operatorname{im}(q_{h'})$$

for the h'-th canonical injection $q_{h'} : \mathcal{R}_{\mathcal{M}}(\operatorname{dom}(h')) \to \mathcal{F}^{i+1}_{\mathcal{M}}(G)$ into the colimit of the gluing diagram $\mathcal{D}^{\mathcal{F}^i_{\mathcal{M}}(G)}$ in the simple case and of the generalized gluing diagram $\mathcal{D}_{\mathcal{F}^i_{\mathcal{M}}(G)}$ in the strict case. Thus the ordered pair $\mathcal{N}^{\mathrm{app}}_G = (Ev^{\mathrm{app}}(G), \succ^{\mathrm{app}}_G)$ is called a *causal net of the rule application events with respect to* G.

For natural numbers $n > 0$ the *restrictions* of $\mathcal{N}^{\mathrm{app}}_G$ to n, denoted by $\mathcal{N}^{\mathrm{app}}_G \upharpoonright n$, are the ordered pairs $(Ev^{\mathrm{app}}(G) \upharpoonright n, \succ^{\mathrm{app}}_G \upharpoonright n)$ with $Ev^{\mathrm{app}}(G) \upharpoonright n = \{(h, i) \in Ev^{\mathrm{app}}(G) \,|\, i \leq n\}$, where $\succ^{\mathrm{app}}_G \upharpoonright n$ is the restriction of $\succ^{\mathrm{app}}_G$ to $Ev^{\mathrm{app}}(G) \upharpoonright n$.

Lemma 3.1. *Machine* $\mathcal{M}^{\mathrm{Sierp}}$ *is such that for every rule application event* (h, i) *with respect to* Δ_0 *with* $i \geq 0$ *there exists a unique ordered triple* (h_1, h_2, h_3) *of elements of* $\mathcal{P}\ell(\mathcal{F}^{i+1}_{\mathcal{M}^{\mathrm{Sierp}}}(\Delta_0))$ *such that the following condition holds*:

$$(\alpha) \qquad (h_j, i + 1) \succ^{\mathrm{app}}_{\Delta_0} (h, i) \text{ and } h_j = f_j \circ h \text{ for all } j \in \{1, 2, 3\},$$

where f_1, f_2, f_3 *form the iteration function system for Sierpiński gasket, cf. Appendix C, and* $\circ$ *denotes the composition of functions.*

Proof. We prove the lemma by induction on i.

Corollary 3.1. *Causal net* $\mathcal{N}^{\mathrm{app}}_{\Delta_0}$ *of the rule application events with respect to* Δ_0 *for machine* $\mathcal{M}^{\mathrm{Sierp}}$ *is isomorphic to the* (*ternary*) *tree* $\mathbb{T}$ *whose vertices are finite strings* (*including empty string*) *of digits in* $\{1, 2, 3\}$*, the edges are ordered pairs* $(\Gamma j, \Gamma)$ *for a finite string* Γ *and a digit* $j \in \{1, 2, 3\}$*, where the graph isomorphism* $\mathrm{iz} : \mathbb{T} \to \mathcal{N}^{\mathrm{app}}_{\Delta_0}$ *is defined inductively by*

- $\mathrm{iz}(\text{empty string}) = (\mathrm{id}_{\Delta_0}, 0)$*, where* id_{Δ_0} *is the identity graph homomorphism on* Δ_0*,*
- $\mathrm{iz}(\Gamma j) = (h', \mathrm{length}(\Gamma) + 1)$ *for a unique* h' *which is the* j*-th element of a unique ordered triple which satisfies the condition* (α) *for that* h *for which* $\mathrm{iz}(h) = (h, \mathrm{length}(\Gamma))$.

Proof. The corollary is a consequence of Lemma 3.1.

Corollary 3.2. *Machine* $\mathcal{M}^{\mathrm{Sierp}}$ *is such that for every rule application event* (h, i) *with respect to* Δ_0 *for* $i \geq 0$ *the unique ordered triple* (h_1, h_2, h_3) *of elements of* $\mathcal{P}\ell(\mathcal{F}^{i+1}_{\mathcal{M}^{\mathrm{Sierp}}}(\Delta_0)$ *satisfying the condition* (α) *for* h *determines a directed multi-hypergraph* $\mathcal{G}_{(h,i)}$ (*see Appendix A*) *whose set of hyperedges is the set*

$\{(h_1, i{+}1), (h_2, i{+}1), (h_3, i{+}1)\}$, *the set of vertices is the union* $\bigcup_{1\leq j\leq 3} V(\mathrm{im}(h_j))$ *and the source and target functions* s, t *are given by*

$$s((h_j, i+1)) = \{h_j((0,0)), h_j((1,0))\},$$
$$t((h_j, i+1)) = \{h_j((\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}))\} \quad \textit{for all } j \in \{1,2,3\}.$$

Moreover, for every rule application event (h, i) *with respect to* Δ_0 *for* $i \geq 0$ *the directed multi-hypergraph* $\mathcal{G}_{(h,i)}$ *is isomorphic to the directed multi-hypergraph* $\mathcal{G}_{(\mathrm{id}_{\Delta_0},0)}$.

Proof. The corollary is a consequence of Lemma 3.1.

Remark 3.1. The directed multi-hypergraph $\mathcal{G}_{(h,i)}$ in Corollary 3.2 could model some interaction between the rule application events in the computation process of $\mathcal{M}^{\mathrm{Sierp}}$ starting with Δ_0 and represented by $\mathcal{N}^{\mathrm{app}}_{\Delta_0}$. This interaction could be a gluing pattern understood as in the main theorem of [16].

Remark 3.2. Since the multi-hyperedge membrane systems $\mathcal{S}^{\mathrm{Sierp}}_n$ in [16] for $n \geq 0$ are aimed to display the self-similar structure of (the iterations of) Sierpiński gasket by using isomorphisms of directed multi-hypergraphs and net $\mathcal{N}^{\mathrm{app}}_{\Delta_0}$ represents the computation process of machine $\mathcal{M}^{\mathrm{Sierp}}$ starting with Δ_0, one can see (in the light of Corollaries 3.1 and 3.2) that the self-similar structure (or form) of the contours of the iterations of Sierpiński gasket coincides[2] with (or simply is) the process of their generation by machine $\mathcal{M}^{\mathrm{Sierp}}$. This coincidence is similar to the coincidence of *Nautilus* shell, illustrated in Fig. 1 in [22], with the process of its growth.

Final Remark 3.3. The author expects that the geometrical G–P–R machines and their extensions to higher dimensions could provide the mathematical foundations for *the atomic basis of biological symmetry and periodicity*[3] due to Antonio Lima-da-Faria [9]. These foundations could explicate the links of cellular automata approach to complexity in biology in S. Wolfram's *A New Kind of Science* with *Evolution without selection* [10] pointed out by B. Goertzel in his review of *A New Kind of Science* in [6].

Open problem One can define geometrical G–P–R machines whose instantaneous descriptions are finite graphs with vertices labelled by multisets and the machine rewriting rules contain multiset rewriting rules like in membrane computing [17].

How to extract in the case of these machines the counterparts of causal nets to be subject of measuring uncertainty via fractal dimension like e.g. in [7], [8].

[2] by Corollaries 3.1 and 3.2 the restrictions $\mathcal{N}^{\mathrm{app}}_{\Delta_0} \restriction n$ together with the directed hypergraphs $\mathcal{G}(h,i)$ provide a construction of multihyperedge membrane systems (with the restrictions $\mathcal{N}^{\mathrm{app}}_{\Delta_0} \restriction n$ as their underlying trees) isomorphic to $\mathcal{S}^{\mathrm{Sierp}}_n$.

[3] selfsimilarity characterized in terms of geometrical G–P–R machines like in $\mathcal{M}^{\mathrm{Sierp}}$ case could be a counterpart of spatial periodicity with respect to both time and scale changes.

Appendix A. Graph-theoretical and category-theoretical preliminaries

A [*finite*] *labelled directed graph over* a set Σ of labels is defined as an ordered triple $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \ell_{\mathcal{G}})$, where $V(\mathcal{G})$ is a [finite] *set of vertices* of $\mathcal{G}$, $E(\mathcal{G})$ is a subset of $V(\mathcal{G}) \times V(\mathcal{G})$ called the *set of edges* of $\mathcal{G}$, and $\ell_{\mathcal{G}}$ is a function from $V(\mathcal{G})$ into Σ called the *labelling function* of $\mathcal{G}$. We drop the adjective 'directed' if there is no risk of confusion.

A *homomorphism of a labelled directed graph* $\mathcal{G}$ *over* Σ *into a labelled directed graph* $\mathcal{G}'$ *over* Σ is an ordered triple $(\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ such that h is a function from $V(\mathcal{G})$ into $V(\mathcal{G}')$ which satisfies the following conditions:

(H_1) $(v, v') \in E(\mathcal{G})$ implies $(\mathrm{h}(v), \mathrm{h}(v')) \in E(\mathcal{G}')$ for all $v, v' \in V(\mathcal{G})$,
(H_2) $\ell_{\mathcal{G}'}(\mathrm{h}(v)) = \ell_{\mathcal{G}}(v)$ for every $v \in V(\mathcal{G})$.

If a triple $h = (\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ is a homomorphism of a labelled directed graph $\mathcal{G}$ over Σ into a labelled directed graph $\mathcal{G}'$ over Σ, we denote this triple by $h : \mathcal{G} \to \mathcal{G}'$, we write $\mathrm{dom}(h)$ and $\mathrm{cod}(h)$ for $\mathcal{G}$ and $\mathcal{G}'$, respectively, according to category theory convention, and we write $h(v)$ for the value $\mathrm{h}(v)$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *embedding of* $\mathcal{G}$ *into* $\mathcal{G}'$, denoted by $h : \mathcal{G} \rightarrowtail \mathcal{G}'$, if the following condition holds:

(E) $h(v) = h(v')$ implies $v = v'$ for all $v, v' \in V(\mathcal{G})$.

An embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ is an *inclusion of* $\mathcal{G}$ *into* $\mathcal{G}'$, denoted by $h : \mathcal{G} \hookrightarrow \mathcal{G}'$, if the following holds:

(I) $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is a *labelled subgraph* of a labelled directed graph $\mathcal{G}'$ over Σ if there exists an inclusion $h : \mathcal{G} \hookrightarrow \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define the *image* of h, denoted by $\mathrm{im}(h)$, to be a labelled directed graph $\widehat{\mathcal{G}}$ over Σ such that $V(\widehat{\mathcal{G}}) = \{h(v) \,|\, v \in V(\mathcal{G})\}$, $E(\widehat{\mathcal{G}}) = \{(h(v), h(v')) \,|\, (v, v') \in E(\mathcal{G})\}$, and the labelling function $\ell_{\hat{\mathcal{G}}}$ of $\widehat{\mathcal{G}}$ is the restriction of the labelling function $\ell_{\mathcal{G}'}$ of $V(\mathcal{G}')$ to the set $V(\widehat{\mathcal{G}})$, i.e., $\ell_{\hat{\mathcal{G}}}(v) = \ell_{\mathcal{G}'}(v)$ for every $v \in V(\widehat{\mathcal{G}})$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *isomorphism* of $\mathcal{G}$ into $\mathcal{G}'$ if there exists a homomorphism $h^{-1} : \mathcal{G}' \to \mathcal{G}$ of labelled directed graphs over Σ, called the inverse of h, such that the following conditons hold:

(Iz_1) $h^{-1}(h(v)) = v$ for every $v \in V(\mathcal{G})$,
(Iz_2) $h(h^{-1}(v)) = v$ for every $v \in V(\mathcal{G}')$.

We say that a labelled directed graph $\mathcal{G}$ over Σ *is isomorphic* to a labelled directed graph $\mathcal{G}'$ over Σ if there exists an isomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define a homomorphism $\dot{h} : \mathcal{G} \to \mathrm{im}(h)$ by $\dot{h}(v) = h(v)$ for every $v \in V(\mathcal{G})$. This homomorphism $\dot{h}$ is an isomorphism of $\mathcal{G}$ into $\mathrm{im}(h)$, called an *isomorphism deduced by* h.

For a labelled directed graph $\mathcal{G}$ over Σ, the *identity homomorphism* (or simply, *identity of* $\mathcal{G}$), denoted by $\mathrm{id}_{\mathcal{G}}$, is the homomorphism $h : \mathcal{G} \to \mathcal{G}$ such that $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is an *isomorphically perfect* labelled directed graph over Σ if the identity homomorphism $\mathrm{id}_{\mathcal{G}}$ is a unique isomorphism of labelled directed graph $\mathcal{G}$ into $\mathcal{G}$.

Lemma A.1. *Let $\mathcal{G}$ be an isomorphically perfect labelled directed graph over Σ and let $h : \mathcal{G} \to \mathcal{G}'$, $h' : \mathcal{G} \to \mathcal{G}'$ be two isomorphisms of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ. Then $h = h'$.*

We say that a set or a class $\mathcal{A}$ of labelled directed graphs over Σ is *skeletal* if for all labelled directed graphs $\mathcal{G}, \mathcal{G}'$ in $\mathcal{A}$ if they are isomorphic, then $\mathcal{G} = \mathcal{G}'$.

A *gluing diagram* $\mathcal{D}$ of labelled directed graphs over Σ is defined by:

— its *set $\mathcal{I}$ of indexes* with a distinguished index $\Delta \in \mathcal{I}$, called the *center* of $\mathcal{D}$,
— its *family* $\mathcal{G}_i$ $(i \in \mathcal{I})$ *of labelled directed graphs* over Σ,
— its *family* gl_i $(i \in \mathcal{I} - \{\Delta\})$ *of gluing conditions* which are sets of ordered pairs such that
  (i) $\mathrm{gl}_i \subseteq V(\mathcal{G}_\Delta) \times V(\mathcal{G}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,
  (ii) $(v, v') \in \mathrm{gl}_i$ implies $\ell_{\mathcal{G}_\Delta}(v) = \ell_{\mathcal{G}_i}(v')$ for all $v \in V(\mathcal{G}_\Delta)$, $v' \in V(\mathcal{G}_i)$, and for every $i \in \mathcal{I} - \{\Delta\}$,
  (iii) for every $i \in \mathcal{I} - \{\Delta\}$ if gl_i is non-empty, then there exists a bijection

$$b_i : L(\mathrm{gl}_i) \to R(\mathrm{gl}_i)$$

for $L(\mathrm{gl}_i) = \{v \mid (v, v') \in \mathrm{gl}_i \text{ for some } v'\}$ and $R(\mathrm{gl}_i) = \{v' \mid (v, v') \in \mathrm{gl}_i \text{ for some } v\}$ such that $\big\{(v, b_i(v)) \mid v \in L(\mathrm{gl}_i)\big\} = \mathrm{gl}_i$.

For a gluing diagram $\mathcal{D}$ of labelled directed graphs over Σ we define a *cocone* of $\mathcal{D}$ to be a family $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of homomorphisms of labelled directed graphs over Σ (here $\mathrm{cod}(h_i) = \mathcal{G}$ for every $i \in \mathcal{I}$) such that

$$h_\Delta(v) = h_i(v')$$

for every pair $(v, v') \in \mathrm{gl}_i$ and every $i \in \mathcal{I} - \{\Delta\}$.

A cocone $q_i : \mathcal{G}_i \to \widetilde{\mathcal{G}}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ is called a *colimiting cocone of* $\mathcal{D}$ if for every cocone $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ there exists a unique homomorphism $h : \widetilde{\mathcal{G}} \to \mathcal{G}$ of labelled directed graphs $\widetilde{\mathcal{G}}, \mathcal{G}$ over Σ such that $h(q_i(v)) = h_i(v)$ for every $v \in V(\mathcal{G}_i)$ and for every $i \in \mathcal{I}$. The labelled directed graph $\widetilde{\mathcal{G}}$ is called a *colimit* of $\mathcal{D}$, the homomorphisms q_i $(i \in \mathcal{I})$ are called *canonical injections* and the unique homomorphism h is called the *mediating morphism* for $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$.

For a gluing diagram $\mathcal{D}$ one constructs its colimit $\widetilde{\mathcal{G}}$ in the following way:

— $V(\widetilde{\mathcal{G}}) = \bigcup_{i\in\mathcal{I}} (V_i \times \{i\})$, where
$V_\Delta = V(\mathcal{G}_\Delta)$ for the center Δ of $\mathcal{D}$,
$V_i = V(\mathcal{G}_i) - R(\mathrm{gl}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,

— $E(\widetilde{\mathcal{G}}) = \bigcup_{i\in\mathcal{I}} E_i$, where
$E_\Delta = \big\{\big((v,\Delta),(v',\Delta)\big) \,|\, (v,v') \in E(\mathcal{G}_\Delta)\big\}$ for the center Δ of $\mathcal{D}$,

$$\begin{aligned}E_i = &\big\{\big((v,i),(v',i)\big) \,|\, (v,v') \in E(\mathcal{G}_i) \text{ and } \{v,v'\} \subseteq V_i\big\}\\ &\cup \big\{\big((v,\Delta),(v',\Delta)\big) \,|\, (v,v'') \in \mathrm{gl}_i,\ (v',v''') \in \mathrm{gl}_i,\\ &\qquad\qquad \text{and } (v'',v''') \in E(\mathcal{G}_i) \text{ for some } v'',v'''\big\}\\ &\cup \big\{\big((v,\Delta),(v',i)\big) \,|\, v' \in V_i,\ (v,v'') \in \mathrm{gl}_i \text{ and } (v'',v') \in E(\mathcal{G}_i) \text{ for some } v''\big\}\\ &\cup \big\{\big((v,i),(v',\Delta)\big) \,|\, v \in V_i,\ (v',v'') \in \mathrm{gl}_i \text{ and } (v,v'') \in E(\mathcal{G}_i) \text{ for some } v''\big\}\end{aligned}$$

for every $i \in \mathcal{I} - \{\Delta\}$,

— the labelling function $\ell_{\widetilde{\mathcal{G}}}$ is defined by $\ell_{\widetilde{\mathcal{G}}}((v,i)) = \ell_{\mathcal{G}_i}(v)$ for every $(v,i) \in V(\widetilde{\mathcal{G}})$.

The definition of a colimiting cocone of a gluing diagram $\mathcal{D}$ provides that any other colimit of $\mathcal{D}$ is isomorphic to the colimit of $\mathcal{D}$ constructed above. Hence one proves the following lemma.

Lemma A.2. *Let $\mathcal{D}$ be a gluing diagram of labelled graphs over Σ. Then for every colimiting cocone $q_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ if $i' \neq i''$, then*

$$\big(V(\mathrm{im}(q_{i'})) - V(\mathrm{im}(q_\Delta))\big) \cap \big(V(\mathrm{im}(q_{i''})) - V(\mathrm{im}(q_\Delta))\big) = \varnothing$$

for all $i', i'' \in \mathcal{I} - \{\Delta\}$, where Δ is the center of $\mathcal{D}$ and the elements of nonempty $V(\mathrm{im}(q_i)) - V(\mathrm{im}(q_\Delta))$ with $i \neq \Delta$ are 'new' elements and the elements of $V(\mathrm{im}(q_\Delta))$ are 'old' elements.

A *generalized gluing diagram* $\mathcal{D}$ of labelled directed graphs over Σ is defined by:

— its *set $\mathcal{I}$ of indexes* with a distinguished index $\Delta \in \mathcal{I}$, called the *center* of $\mathcal{D}$,

— its *family* $\mathcal{G}_i$ $(i \in \mathcal{I})$ *of labelled directed graphs* over Σ,

— its *family* gl^i_j $((i,j)) \in \mathcal{I} \times (\mathcal{I} - \{\Delta\})$ and $i \neq j)$ *of gluing conditions* which are such that
- the set $\mathcal{I}^\Delta = \mathcal{I}$ with families $\mathcal{G}_i$ $(i \in \mathcal{I})$ and gl^Δ_i $(i \in \mathcal{I} - \{\Delta\})$ form a gluing diagram $\mathcal{D}^\Delta$ with Δ as the center of $\mathcal{D}^\Delta$,
- for every $i \in \mathcal{I} - \{\Delta\}$ the set $I^i = \mathcal{I} - \{\Delta\}$ with families $\mathcal{G}_i$ $(i \in \mathcal{I} - \{\Delta\})$ and gl^i_j $(j \in \mathcal{I} - \{i, \Delta\})$ form a gluing diagram $\mathcal{D}^i$ with i as the center for $\mathcal{D}^i$,
- the following conditions hold:

(G_1) $R(\mathrm{gl}^\Delta_i) \cap L(\mathrm{gl}^i_j) = \varnothing$ for all i, j with $\{i,j\} \subset \mathcal{I} - \{\Delta\}$ and $i \neq j$,

(G_2) $(\mathrm{gl}^i_j)^{-1} = \mathrm{gl}^j_i$ for all i, j with $\{i,j\} \subset \mathcal{I} - \{\Delta\}$ and $i \neq j$, where for $Q \subset A \times B$

$$(Q)^{-1} = \big\{(x,y) \in B \times A \,|\, (y,x) \in A \times B\big\}.$$

For a generalized gluing diagram $\mathcal{D}$ of labelled directed graphs over Σ we define a *cocone* of $\mathcal{D}$ to be a family $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of homomorphisms of labelled directed graphs over Σ (here $\mathrm{cod}(h_i) = \mathcal{G}$ for every $i \in \mathcal{I}$) such that for every $i \in \mathcal{I}$ the sub-family $h_j : \mathcal{G}_j \to \mathcal{G}$ $(j \in \mathcal{I}^i)$ is a cocone of the diagram $\mathcal{D}^i$.

For a generalized gluing diagram $\mathcal{D}$ a *colimiting cocone* of $\mathcal{D}$, a *colimit* of $\mathcal{D}$, the *canonical injections*, and the *mediating morphism* are defined in the same way as for a gluing diagram, e.g. a cocone $q_i : \mathcal{G}_i \to \widetilde{\mathcal{G}}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ is called a *colimiting cocone of* $\mathcal{D}$ if for every cocone $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ there exists a unique homomorphism $h : \widetilde{\mathcal{G}} \to \mathcal{G}$ of labelled directed graphs $\widetilde{\mathcal{G}}, \mathcal{G}$ over Σ such that $h(q_i(v)) = h_i(v)$ for every $v \in V(\mathcal{G}_i)$ and for every $i \in \mathcal{I}$.

Lemma A.3. *Let $\mathcal{D}$ be a generalized gluing diagram with finite set $\mathcal{I}$ of its indexes and with center Δ, such that the following condition holds*:

(G_3) *for all $i, i', j \in \mathcal{I} - \{\Delta\}$ if $i \neq i'$, then $L(\mathrm{gl}_i^j) \cap L(\mathrm{gl}_{i'}^j) = \varnothing$.*

Then one constructs a colimit of $\mathcal{D}$ to be a labelled directed graph $\widetilde{\mathsf{G}}$ which is determined by an arbitrary nonrepetitive sequence $i_1, \ldots, i_{n_0}$ of elements of $\mathcal{I} - \{\Delta\} = \{i_1, \ldots, i_{n_0}\}$ and which is defined in the following way:

— $V(\widetilde{\mathsf{G}}) = \bigcup_{i \in \mathcal{I}} (V_i \times \{i\})$, *where* $V_\Delta = V(\mathcal{G}_\Delta)$, $V_{i_1} = V(\mathcal{G}_{i_1}) - R(\mathrm{gl}_{i_1}^\Delta)$, *for every k with* $1 < k \leq n_0$

$$V_{i_k} = V(\mathcal{G}_{i_k}) - \Big(R(\mathrm{gl}_{i_k}^\Delta) \cup \bigcup_{1 \leq m < k} L(\mathrm{gl}_{i_m}^{i_k}) \Big),$$

— $E(\widetilde{\mathsf{G}}) = \bigcup_{i \in \mathcal{I}} E_i$, *where* $E_\Delta = \big\{ ((v, \Delta), (v', \Delta)) \,|\, (v, v') \in E(\mathcal{G}_\Delta) \big\}$,
for every $i \in \mathcal{I} - \{\Delta\}$
$E_i = E_i^1 \cup E_i^2 \cup E_i^3 \cup E_i^4$ *for*

$E_i^1 = \big\{ ((v, i), (v', i)) \,|\, \{(v, i), (v', i)\} \subset V(\widetilde{\mathsf{G}})$ *and* $(v, v') \in E(\mathcal{G}_i) \big\}$,

$E_i^2 = \big\{ ((v, k), (v', j)) \,|\, \{(v, k), (v', j)\} \subset V(\widetilde{\mathsf{G}}),\ i \notin \{k, j\} \subset \mathcal{I},$
$\quad (v, v'') \in \mathrm{gl}_i^k,\ (v', v''') \in \mathrm{gl}_i^j,$ *and* $(v'', v''') \in E(\mathsf{G}_i)$ *for some* $v'', v''' \big\}$,

$E_i^3 = \big\{ ((v, i), (v', j)) \,|\, \{(v, i), (v', j)\} \subset V(\widetilde{\mathsf{G}}),\ i \neq j \in \mathcal{I},$
$\quad (v', v'') \in \mathrm{gl}_i^j,$ *and* $(v, v'') \in E(\mathcal{G}_i)$ *for some* $v'' \big\}$,

$E_i^4 = \big\{ ((v, j), (v', i)) \,|\, \{(v, j), (v', i)\} \subset V(\widetilde{\mathsf{G}}),\ i \neq j \in \mathcal{I},$
$\quad (v, v'') \in \mathrm{gl}_i^j,$ *and* $(v'', v') \in E(\mathcal{G}_i)$ *for some* $v'' \big\}$,

— *the labelling function $\ell_{\widetilde{\mathsf{G}}}$ is defined by $\ell_{\widetilde{\mathsf{G}}}((v, i)) = \ell_{\mathcal{G}_i}(v)$ for every $(v, i) \in V(\widetilde{\mathsf{G}})$.*

Proof. Since by (G_3) for all $i \in \mathcal{I} - \{\Delta\}$ and $v \in V(\mathcal{G}_i) - V_i$ there exists a unique ordered pair $(v^*, i^*) \in V(\widetilde{\mathsf{G}})$ such that $(v^*, v) \in \mathrm{gl}_i^{i^*}$, one defines the i-th component $q_i : \mathcal{G}_i \to \widetilde{\mathsf{G}}$ $(i \in \mathcal{I} - \{\Delta\})$ of colimiting cocone by

$$q_i(v) = \begin{cases} (v, i) & \text{if } v \in V_i, \\ (v^*, i^*) & \text{otherwise.} \end{cases}$$

□

Lemma A.4. *Let $\mathcal{D}$ be a generalized gluing diagram with finite set $\mathcal{I}$ of its indexes and with center Δ, such that the condition (G_3) holds and let $q_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ be a colimiting cocone of $\mathcal{D}$. Then for every $H \subseteq \mathcal{I} - \{\Delta\}$ if*

$$\bigcap_{i \in H} \big(V(\mathrm{im}(q_i)) - V(\mathrm{im}(q_\Delta))\big) \neq \varnothing,$$

then H has at most two elements and if $H = \{i, i'\}$ with $i \neq i'$, then $\mathrm{gl}^i_{i'}$ is nonempty.

Proof. The lemma is a consequence of Lemma A.3 and the fact that two different colimits of a generalized gluing diagram are always isomorphic labelled graphs.

For two directed graphs $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$, we define their *union* by

$$G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)).$$

We introduce the following new concepts.

By a *directed multi-hypergraph* we mean a structure $\mathcal{G}$ given by its *set* $E(\mathcal{G})$ *of hyperedges*, its *set* $V(\mathcal{G})$ *of vertices* and the *source* and *target* mappings

$$s_{\mathcal{G}} : E(\mathcal{G}) \to \mathcal{P}(V(\mathcal{G})), \quad t_{\mathcal{G}} : E(\mathcal{G}) \to \mathcal{P}(V(\mathcal{G}))$$

such that $V(\mathcal{G})$ together with

$$\big\{(\mathcal{V}_1, \mathcal{V}_2) \,|\, s_{\mathcal{G}}(e) = \mathcal{V}_1 \text{ and } t_{\mathcal{G}}(e) = \mathcal{V}_2 \text{ for some } e \in E(\mathcal{G})\big\}$$

form a directed hypergraph as in [5], where $\mathcal{P}(X)$ denotes the set of all subsets of a set X.

We say that two directed multi-hypergraphs $\mathcal{G}, \mathcal{G}'$ are *isomorphic* if there exist two bijections $h : V(\mathcal{G}) \to V(\mathcal{G}')$, $h' : E(\mathcal{G}) \to E(\mathcal{G}')$ such that

$$s_{\mathcal{G}'}(h'(e)) = \{h(v) \,|\, v \in s_{\mathcal{G}}(e)\} \text{ and } t_{\mathcal{G}'}(h'(e)) = \{h(v) \,|\, v \in t_{\mathcal{G}}(e)\}$$

for all $e \in E(\mathcal{G})$.

Appendix B

We recall an idea of a Gandy–Păun–Rozenberg machine, briefly G–P–R machine, introduced in [14].

The core of a G–P–R machine is a finite set of rewriting rules for certain finite directed labelled graphs, where these graphs are instantenous descriptions for the computation process realized by the machine.

The conflictless parallel (simultaneous) application of the rewriting rules of a G–P–R machine is realized in Gandy's machine mode (according to Local Causation Principle, cf. [3]), where (local) maximality of "causal neighbourhoods" replaces (global) maximality of, e.g. conflictless set of evolution rules applied simultaneously to a membrane structure which appears during the evolution process generated by a P system [17]. Therefore one can construct a Gandy's machine from a G–P–R machine in an immediate way, see [14].

For all unexplained terms and notation of category theory and graph theory we refer the reader to Appendix A.

Definition B.1. A G–P–R *machine* $\mathcal{M}$ is determined by the following data:

— a finite set $\Sigma_{\mathcal{M}}$ of labels or symbols of $\mathcal{M}$,
— a skeletal set $\mathcal{S}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over Σ, which are called *instantenous descriptions* of $\mathcal{M}$,
— a function $\mathcal{F}_{\mathcal{M}} : \mathcal{S}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}}$ called the *transition function* of $\mathcal{M}$,
— a function $\mathcal{R}_{\mathcal{M}} : \mathrm{PREM}_{\mathcal{M}} \to \mathrm{CONCL}_{\mathcal{M}}$ from a finite skeletal set $\mathrm{PREM}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ onto a finite skeletal set $\mathrm{CONCL}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ such that $\mathcal{R}_{\mathcal{M}}$ determines the *set*

$$\widetilde{\mathcal{R}}_{\mathcal{M}} = \{P \vdash C \mid P \in \mathrm{PREM}_{\mathcal{M}} \text{ and } C = \mathcal{R}_{\mathcal{M}}(P)\}$$

of rewriting rules of $\mathcal{M}$ which are identified with ordered pairs $r = (P_r, C_r)$, where the graph $P_r \in \mathrm{PREM}_{\mathcal{M}}$ is the *premise* of r and the graph $C_r = \mathcal{R}_{\mathcal{M}}(P_r)$ is the *conclusion* of r,
— a subset $\mathcal{I}_{\mathcal{M}}$ of $\mathcal{S}_{\mathcal{M}}$ which is the set of *initial instantaneous descriptions of* $\mathcal{M}$.

The above data are subject of the following conditions:

1) $V(\mathcal{G}) \subseteq V(\mathcal{F}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$,
2) $V(\mathcal{G}) \subseteq V(\mathcal{R}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathrm{PREM}_{\mathcal{M}}$,
3) the rewriting rules of $\mathcal{M}$ are *applicable* to $\mathcal{S}_{\mathcal{M}}$ which means that for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$ the set

$$\begin{aligned}\mathcal{P}\ell(\mathcal{G}) = \big\{h \mid\ & h \text{ is an embedding of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h) \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h) = \mathcal{G} \\ & \text{such that for every embedding } h' \text{ of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h') \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h') = \mathcal{G} \\ & \text{if } \mathrm{im}(h) \text{ is a labelled subgraph of } \mathrm{im}(h'), \text{ then } h = h'\big\}\end{aligned}$$

of *maximal applications*[4] h of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ of $\mathcal{M}$ in places $\mathrm{im}(h)$ is such that the following conditions hold:
(i) $V(\mathcal{G}) = \bigcup\limits_{h \in \mathcal{P}\ell(\mathcal{G})} V(\mathrm{im}(h))$, $E(\mathcal{G}) = \bigcup\limits_{h \in \mathcal{P}\ell(\mathcal{G})} E(\mathrm{im}(h))$,

[4] with respect to the relation of being a labelled subgraph which can be treated as a natural priority relation between the applications of the rewriting rules

(ii) for all $h_1, h_2 \in \mathcal{P}\ell(\mathcal{G})$ the equation $\ell_{\mathcal{G}_{h_1}}(\dot{h}_1^{-1}(v)) = \ell_{\mathcal{G}_{h_2}}(\dot{h}_2^{-1}(v))$ holds for every $v \in V(\mathrm{im}(h_1)) \cap V(\mathrm{im}(h_2))$, where $\ell_{\mathcal{G}_{h_1}}$, $\ell_{\mathcal{G}_{h_2}}$ are the labelling functions of $\mathcal{G}_{h_1} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_1))$, $\mathcal{G}_{h_2} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_2))$, respectively, and $\dot{h}_1^{-1}$, $\dot{h}_2^{-1}$ are the inverses of isomorphisms induced by the embeddings h_1, h_2, respectively.

(iii) $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is a colimit of a gluing diagram $\mathcal{D}^{\mathcal{G}}$ constructed in the following way (the construction of $\mathcal{D}^{\mathcal{G}}$ is provided by (ii)):
- the set $\mathcal{I}$ of indexes of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{I} = \mathcal{P}\ell(\mathcal{G}) \cup \{\mathit{\Delta}\}$, where $\mathit{\Delta} \notin \mathcal{P}\ell(\mathcal{G})$ is the center of $\mathcal{D}^{\mathcal{G}}$,
- the family $\mathcal{G}_i$ $(i \in \mathcal{I})$ of labelled graphs of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{G}_h = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ for every $h \in \mathcal{P}\ell(\mathcal{G})$, and $\mathcal{G}_{\mathit{\Delta}}$ is such that $V(\mathcal{G}_{\mathit{\Delta}}) = V(\mathcal{G})$, $E(\mathcal{G}_{\mathit{\Delta}}) = \varnothing$, and the labelling function $\ell_{\mathcal{G}_{\mathit{\Delta}}}$ is such that provided by (ii)
$$\ell_{\mathcal{G}_{\mathit{\Delta}}}(v) = \ell_{\mathcal{G}_h}(\dot{h}^{-1}(v))$$
for every $v \in V(\mathrm{im}(h))$ and every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by the embedding h,
- the gluing conditions gl_h $(h \in \mathcal{P}\ell(\mathcal{G}))$ of $\mathcal{D}^{\mathcal{G}}$ are defined by
$$\mathrm{gl}_h = \big\{(v, \dot{h}^{-1}(v)) \,|\, v \in V(\mathrm{im}(h))\big\}$$
for every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by embedding h,

(iv) the following equations hold:
$$V(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i \in \mathcal{I}} V(\mathrm{im}(q_i))$$
$$\text{and } E(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i \in \mathcal{I}} E(\mathrm{im}(q_i))$$
for the canonical injections $q_i : \mathcal{G}_i \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ $(i \in \mathcal{I})$ forming a colimiting cocone of the diagram $\mathcal{D}^{\mathcal{G}}$ defined in (iii),

(v) the canonical injection $q_{\mathit{\Delta}} : \mathcal{G}_{\mathit{\Delta}} \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is an inclusion of labelled graphs, where $\mathit{\Delta}$ is the center of $\mathcal{D}^{\mathcal{G}}$ and $q_{\mathit{\Delta}}$ is $\mathit{\Delta}$-th element of the colimiting cocone in (iv).

Thus $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is the result of simultaneous application of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ in the places $\mathrm{im}(h)$ for $h \in \mathcal{P}\ell(\mathcal{G})$, where one replaces simultaneously $\mathrm{im}(h)$ by $\mathrm{im}(q_h)$ in $\mathcal{G}$ for $h \in \mathcal{P}\ell(\mathcal{G})$, respectively.

A finite sequence $\big(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})\big)_{i=0}^n$ is called a *finite computation of* $\mathcal{M}$, the number n is called the *time* of this computation, and $\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})$ is called the *final instantaneous description* for this computation if
$$\mathcal{F}^0_{\mathcal{M}}(\mathcal{G}) = \mathcal{G} \in \mathcal{I}_{\mathcal{M}}, \quad \mathcal{F}^{n-1}_{\mathcal{M}}(\mathcal{G}) \neq \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}), \quad \text{and } \mathcal{F}_{\mathcal{M}}(\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})) = \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}),$$
where $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})$ is defined inductively: $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G}) = \mathcal{F}_{\mathcal{M}}\big(\mathcal{F}^{i-1}_{\mathcal{M}}(\mathcal{G})\big)$.

For a computation $\big(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})\big)_{i=0}^n$ its *space* is defined by
$$\mathrm{space}(\mathcal{M}, \mathcal{G}) = \max\{\text{the number of elements of } V(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})) \,|\, 0 \le i \le n\}$$

for $\mathcal{G} \in \mathcal{I}_{\mathcal{M}}$, where intuitively space$(\mathcal{M}, \mathcal{G})$ is understood as the size of hardware measured by the number of indecomposable processors[5] used in the computations.

We recall the following definition from [15].

Definition B.2. A *generalized G–P–R machine* $\mathcal{M}$ is defined by the following data:

— the sets $\Sigma_{\mathcal{M}}$, $\mathcal{S}_{\mathcal{M}}$, $\mathcal{I}_{\mathcal{M}}$ and the functions $\mathcal{R}_{\mathcal{M}} : \mathrm{PREM}_{\mathcal{M}} \to \mathrm{CONCL}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}} : \mathcal{S}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}}$, where $\mathcal{S}_{\mathcal{M}}$, $\mathrm{PREM}_{\mathcal{M}}$, $\mathrm{CONCL}_{\mathcal{M}}$ are skeletal sets of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$, the sets $\Sigma_{\mathcal{M}}$, $\mathrm{PREM}_{\mathcal{M}}$, $\mathrm{CONCL}_{\mathcal{M}}$ are finite sets, the condition 2) holds for $\mathcal{R}_{\mathcal{M}}$, and $\mathcal{I}_{\mathcal{M}}$ is a subset of $\mathcal{S}_{\mathcal{M}}$;

— besides the function $\mathcal{R}_{\mathcal{M}}$ defining *rewriting rules* there is enclosed a new function $\mathcal{R}^a_{\mathcal{M}} : \mathrm{PREM}^a_{\mathcal{M}} \to \mathrm{CONCL}^a_{\mathcal{M}}$, where $\mathrm{PREM}^a_{\mathcal{M}}$, $\mathrm{CONCL}^a_{\mathcal{M}}$ are finite skeletal sets of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ and $\mathcal{R}^a_{\mathcal{M}}$ defines *auxiliary gluing rules* $P \overset{a}{\vdash} C$ ($P \in \mathrm{PREM}^a_{\mathcal{M}}$, $C = \mathcal{R}^a_{\mathcal{M}}(P)$) for defining common parts of the boundaries of new compartments appearing in a step of an evolution process;

— the above data are subject of the following conditions:

A) for every $\mathcal{G} \in \mathrm{PREM}^a_{\mathcal{M}}$ we have $V(\mathcal{G}) \subseteq V(\mathcal{R}^a_{\mathcal{M}}(\mathcal{G}))$, the set $\mathcal{P}\ell(\mathcal{G})$ defined as in 3) satisfies 3)(ii), and there exists a generalized gluing diagram $\mathcal{D}_{\langle\mathcal{G}\rangle}$, called *gluing pattern determined by* $\mathcal{G}$, such that

a_1) the set $\mathcal{I}_{\langle\mathcal{G}\rangle}$ of indexes of $\mathcal{D}_{\langle\mathcal{G}\rangle}$ is a set $\{\Delta\} \cup \dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$ with Δ being the center of $\mathcal{D}_{\langle G\rangle}$, $\dot{\mathcal{I}}_{\langle\mathcal{G}\rangle} \subseteq \mathcal{P}\ell(\mathcal{G})$, and $\Delta \notin \dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$;

a_2) the family of graphs $\mathcal{G}_i$ ($i \in \mathcal{I}_{\langle\mathcal{G}\rangle}$) of $\mathcal{D}_{\langle\mathcal{G}\rangle}$ is such that $V(\mathcal{G}_\Delta) = V(\mathcal{G})$, $E(\mathcal{G}_\Delta) = \varnothing$, and $\mathcal{G}_h = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ for $h \in \dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$;

a_3) the gluing conditions gl^Δ_i ($i \in \dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$) are such that $\mathrm{gl}^\Delta_i = \mathrm{gl}_i$ for gl_i defined as in 3)(iii) for the gluing diagram $\mathcal{D}^{\mathcal{G}}$;

a_4) $\mathcal{R}^a_{\mathcal{M}}(\mathcal{G})$ is a colimit of $\mathcal{D}_{\langle\mathcal{G}\rangle}$ with gluing conditions gl^i_j ($\{i, j\} \subseteq \dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$ and $i \neq j$) such that they are unique together with $\dot{\mathcal{I}}_{\langle\mathcal{G}\rangle}$ to make $\mathcal{R}^a_{\mathcal{M}}(\mathcal{G})$ a colimit of $\mathcal{D}_{\langle\mathcal{G}\rangle}$;

B) for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$ the following conditions hold:

b_1) for $\mathcal{P}\ell^a(\mathcal{G})$ defined as in 3) with $\mathrm{PREM}_{\mathcal{M}}$ replaced by PREM^a_M and for every $h \in \mathcal{P}\ell^a(\mathcal{G})$ and gluing pattern $\mathcal{D}_{\langle\mathrm{dom}(h)\rangle}$ determined by $\mathrm{dom}(h)$ the set $\mathrm{SCP}_h = \{h \circ h' \,|\, h' \in \dot{\mathcal{I}}_{\langle\mathrm{dom}(h)\rangle}\}$, called the *scope of gluing pattern* $\mathcal{D}_{\langle\mathrm{dom}(h)\rangle}$ *in place* h, is a subset of $\mathcal{P}\ell(\mathcal{G})$ defined as in 3) for $\mathcal{G}$ and $\mathrm{PREM}_{\mathcal{M}}$, where $\circ$ denotes the composition of homomorphisms of graphs;

b_2) the set $\mathcal{P}\ell(\mathcal{G})$ defined in 3) satisfies conditions 3)(i), (ii);

b_3) the graph $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is a colimit of a generalized gluing diagram $\mathcal{D}_{\mathcal{G}}$ such that

[5] The indecomposable processors coincide with urelements appearing in those Gandy machines which represent G–P–R machines in [14].

(β_1) the set $\mathcal{I}$ of indexes of $\mathcal{D}_{\mathcal{G}}$ is the same as the set of indexes of $\mathcal{D}^{\mathcal{G}}$ given in 3)(iii), i.e. $\mathcal{I} = \mathcal{P}\ell(\mathcal{G}) \cup \{\Delta\}$,
(β_2) the family of graphs $\mathcal{G}_i$ ($i \in \mathcal{I}$) of $\mathcal{D}_{\mathcal{G}}$ is the same as of $\mathcal{D}^{\mathcal{G}}$ defined in 3)(iii),
(β_3) the gluing condition gl_i^{Δ} is gl_i defined in 3)(iii) for every $i \in \mathcal{I} - \{\Delta\}$,
(β_4) for all h_1, h_2 with $\{h_1, h_2\} \subseteq \mathcal{I} - \{\Delta\}$ and $h_1 \neq h_2$ if there exists $h \in \mathcal{P}\ell^a(\mathcal{G})$ for which $\{h_1, h_2\} \subseteq \mathrm{SCP}_h$, then the gluing condition $\mathrm{gl}_{h_2}^{h_1}$ of $\mathcal{D}_{\mathcal{G}}$ is the gluing condition $\mathrm{gl}_{h'_2}^{h'_1}$ of the gluing pattern determined by $\mathrm{dom}(h)$ for h'_1, h'_2 such that $h \circ h'_1 = h_1$ and $h \circ h'_2 = h_2$,
(β_5) if there does not exist $h \in \mathcal{P}\ell^a(\mathcal{G})$ such that $\{h_1, h_2\} \subseteq \mathrm{SCP}_h$ for h_1, h_2 as in (β_4), then the gluing condition $\mathrm{gl}_{h_2}^{h_1}$ of $\mathcal{D}_{\mathcal{G}}$ is defined to be the empty set;

b_4) the colimiting cocone $q_i : \mathcal{G}_i \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ ($i \in \mathcal{I}$) of $\mathcal{D}_{\mathcal{G}}$ is such that
(β_6) the conditions 3)(iv) and (v) hold with $\mathcal{D}^{\mathcal{G}}$ replaced by $\mathcal{D}_{\mathcal{G}}$,
(β_7) for every at least two element subset H of $\mathcal{I} - \{\Delta\}$ such that

$$\bigcap_{i \in H} \Big(V(\mathrm{im}(q_i)) - V(\mathrm{im}(q_{\Delta})) \Big) \neq \varnothing$$

there exists $h \in \mathcal{P}\ell^a(\mathcal{G})$ such that H is a subset of SCP_h of gluing pattern determined by $\mathrm{dom}(h)$.

The gluing conditions gl_j^i of $\mathcal{D}_{\mathcal{G}}$ defined in (β_4), (β_5) determine common parts of the boundaries of new compartments appearing in a step of an evolution process.

Appendix C

Basing on [18] we present the iterated function systems whose attractors are Koch curve and Sierpiński gasket, respectively. These iterated function systems consist of the bijections from $\mathbb{R}^2$ onto $\mathbb{R}^2$ ($\mathbb{R}^2$ denotes the set of ordered pairs of real numbers) described in terms of matrices as follows:

— for Koch curve

$$f_1^{\mathrm{Koch}}(\mathbf{x}) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{x} \qquad \text{scale by } 1/3$$

$$f_2^{\mathrm{Koch}}(\mathbf{x}) = \begin{bmatrix} 1/6 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \qquad \text{scale by } 1/3\text{, rotate by } 60^\circ$$

$$f_3^{\mathrm{Koch}}(\mathbf{x}) = \begin{bmatrix} 1/6 & \sqrt{3}/6 \\ -\sqrt{3}/6 & 1/6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} \qquad \text{scale by } 1/3\text{, rotate by } -60^\circ$$

$$f_4^{\mathrm{Koch}}(\mathbf{x}) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \qquad \text{scale by } 1/3$$

— for Sierpiński gasket

$$f_1^{\mathrm{Sierp}}(\mathbf{x}) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{x} \qquad \text{scale by } 1/2$$

$$f_2^{\mathrm{Sierp}}(\mathbf{x}) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \qquad \text{scale by } 1/2$$

$$f_3^{\mathrm{Sierp}}(\mathbf{x}) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix} \qquad \text{scale by } 1/2$$

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