An Application of a Reflection Principle

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Abstract

We define a recursive theory which axomatizes a class of models of $I\Delta_0 + \Omega_3 + \neg exp$ all of which share two features: firstly, the set of Δ_0 definable elements of the model is majorized by the set of elements definable by Δ_0 formulae of fixed complexity; secondly, Σ_1 truth about the model is recursively reducible to the set of true Σ_1 formulae of fixed complexity.

In the present paper, we define a consistent recursive theory T, implying $I\Delta_0$ and inconsistent with $I\Delta_0 + exp$, which has the following two properties:

1) in every model $\mathbf{M} \models T$ elements definable by Δ_0 formulae of fixed quantifier complexity are cofinal among all Δ_0 definable elements;

2) for every model $\mathbf{M} \models T$, the set of Σ_1 sentences true in $\mathbf{M}$ is recursively reducible to the set of true Σ_1 sentences whose Δ_0 part has fixed quantifier complexity.

Thus, T axiomatizes to some extent the phenomenon of the cofinality of elements definable by Δ_0 formulae with fixed complexity among all Δ_0 definable elements, and of the reducibility of the set of true Σ_1 sentences to the set of true Σ_1 sentences whose complexity is fixed.

¿From the logical point of view, the idea behind the construction of T seems to be interesting in itself. The axioms of T reduce the validity of a Π_1 sentence ψ to the validity a sentence expressing (roughly) a form of "consistency" of ψ. To show the consistency of T, we have to be able to build a model in which all "consistent" Π_1 sentences are true.

We construct such a model by iterating the following procedure: given a model $\mathbf{M}$ satisfying the "consistency" of the Π_1 sentence ψ_0, we build another model $\mathbf{M}_0$ satisfying ψ_0, and still satisfying the "consistency" of ψ_0. We then move on to the next Π_1 sentence, ψ_1. To carry on the construction, we now must — if $\mathbf{M_0}$ satisfies the "consistency" of ψ_1 — be able to construct another model $\mathbf{M}_1$ satisfying ψ_1, but still satisfying ψ_0 and the "consistency" of ψ_0 and ψ_1. Etc.

Thus, we need our models have the property that "what is true is consistent". Moreover, this property has to be preserved under the iteration. Therefore, what we need is in fact the " "consistency" of the set of true Π_1 and Σ_1 sentences together with the "consistency" of the set of true Π_1 and Σ_1 sentences together with the "consistency" of the set of true Π_1 and Σ_1 sentences...". To make this formal, we have to define a kind of "self–reproducing consistency statement". This is subtle since we are very close to contradicting Gödel's second incompleteness theorem.

The paper is organized as follows. Section 1 is preliminary. Section 2 discusses our basic technical tool: evaluations on sequences of terms. In section 3, we define our "self–reproducing consistency statement", and we argue that it is a kind of reflection principle. Finally, in section 4 we introduce the theory T and prove our main results.

1 Preliminaries

Some notational conventions:

The symbol *log* stands for the discrete–valued binary logarithm function; $exp(x)$ is 2^x. Whenever f denotes a function, $f^{(k)}$ denotes f iterated k times. For a model $\mathbf{M}$, $log^{(k)}(\mathbf{M})$ (the *k-th logarithm* of $\mathbf{M}$) consists of those elements of $\mathbf{M}$ for which $exp^{(k)}$ exists. The variable i, possibly with indices, always ranges over elements of $log^{(3)}$, and the variable j, possibly with indices, ranges over elements of $log^{(4)}$. A "bar" (as in, say, "$\bar{x}$") always denotes a tuple — depending on the context, it may happen that tuples of nonstandard length are also allowed.

We adopt the coding of sets and sequences in bounded arithmetic developed in [HP]. Also the notion of length $lh(\Lambda)$ of a sequence Λ is the one defined in [HP] for bounded arithmetic. If $\Lambda = \langle t_1, \dots, t_l \rangle$ is a sequence of length $l \in log(\mathbf{M})$, then functions from Λ into $\{0, 1\}$ may be coded as subsets of size $lh(\Lambda)$ of $\Lambda \times \{0, 1\}$ (see [S]). We use a somewhat different coding, letting $f : \Lambda \longrightarrow \{0, 1\}$ be represented by the pair $\langle \Lambda, p \rangle$, where p is a function from $\{1, \dots, l\}$ into $\{0, 1\}$ — thus, an object of size $exp(l)$ — with $p(i)$ intended to code $f(t_i)$. Whenever Λ is fixed, we may simply identify f with p.

Our base language L contains the individual constants 0, 1, and the *relational* symbols $+$, $\leq$, $\times$, $|\cdot|$, $\#_2$, $\#_3$, and $\#_4$.

The intuitive meaning of $|x| = y$ is that y is the length of the binary representation of x (equal to $\lceil log(x+1) \rceil$). The $\#_i$'s are to stand for the graphs of the first three smash functions: $x \#_2 y = exp(|x| \cdot |y|)$, $x \#_{n+1} y = exp(|x| \#_n |y|)$ for $n \geq 2$. A hierarchy of functions related to the smash functions is defined by: $\omega_1(x) = x^{|x|}; \omega_{n+1}(x) = exp(\omega_n(|x|))$. Note that for any $n \geq 1$, $\omega_n(x)$ is roughly $x \#_{n+1} x$.

We assume that some appropriate Gödel numbering of L–formulae has been fixed; we shall identify the formulae with their Gödel numbers.

An L–formula φ is in *negation normal form* if no quantifiers in φ occur in the scope of a negation. φ is Δ_0 if all the quantifiers in φ are bounded, i.e. of the form $\exists x \leq y$. Σ_1 and Π_1 formulae are defined in the natural way.

For any natural number r, the class E_r consists of Δ_0 formulae in prenex normal form which contain $(r-1)$ alternations of quantifier blocks, starting with an existential block, and *not* counting sharply bounded quantifiers[1]. The class U_r is defined dually. The class $\exists_r$ consists of Σ_1 formulae of the form $\exists x \psi$ where ψ is U_{r-1}. The class $\forall_r$ is defined dually.

We take $I\Delta_0 + \Omega_3$ to be the theory which consists of: a finite number of basic axioms relating the interpretations of the L–symbols to each other; the induction scheme for all Δ_0 formulae; and an axiom stating that $\#_4$ is a total function (note that this is equivalent to the totality of ω_3). $I\Delta_0 + \Omega_n$, for $i = 1, 2$, is defined analogously. $I\Delta_0$ states only the totality of $+$ and $\times$. $I\Delta_0 + exp$, on the other hand, additionally states the totality of the exp function.

$I\Delta_0^*$ is an auxiliary system which contains the basic axioms and the Δ_0 induction scheme, but no axioms stating the totality of $+, \times$ etc. Thus, a model of $I\Delta_0^*$ may have a greatest element. Note that (under a reasonable choice of the basic axioms), all axioms of $I\Delta_0^*$ are Π_1.

One benefit of working with a relational language is that definining the relativization of a formula poses no difficulties. Namely, if φ is an L–formula, then φ^x is defined inductively, with only the quantifier step non–trivial: $(\exists y \psi)^x := \exists y \leq x \; \psi^x$.

The language L_T is an extension of L obtained by adding function symbols s^φ for all L–formulae φ in negation normal form which begin with an existential quantifier. The intention is that the symbol s^φ stands for a Skolem function for the first existential quantifier in φ. That is, given an L–formula $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$ in negation normal form, s^φ is a function symbol of arity

[1]The notion of *sharply bounded quantifier* is an obvious variant of the one known from functional languages for bounded arithmetic, e.g. in $\forall x \forall y \leq x \exists z \leq x((y = |x| \Rightarrow z = y) \wedge \ldots)$ the quantifier $\forall z$ is sharply bounded.

$1 + lh(\bar{x})$, and $s^\varphi(\bar{t})$ is intended to be some y which satisfies $\psi(\bar{t}, y)$, if such a y exists.

Whenever we speak of a formula $\varphi(\bar{t})$, it is assumed that $\varphi(\bar{x})$ itself is an L–formula, although the terms $\bar{t}$ do not have to be terms of L.

We have to encode the language L_T in arithmetic. We use numbers divisible by 3 to enumerate terms of the form $s^\varphi(\bar{t})$, numbers congruent to 1 (mod 3) for a special enumeration of numerals, and numbers congruent to 2 (mod 3) to enumerate some additional terms. In more detail: we let the number $3\langle\varphi(\bar{x}), \bar{t}\rangle$ correspond to $s^\varphi(\bar{t})$; we let $3k+1$ correspond to a numeral for k ($3k+1$ will be referred to as $\underline{k}$); finally, we let $3k+2$ correspond to a special term s_k (the role of the s_k's is explained by clause (v) of definition 2.3). We also code $\varphi(\bar{t})$ by the ordered pair $\langle\varphi(\bar{x}), \bar{t}\rangle$.

¿From now on, we identify the terms of L_T with their numbers.

The models $\mathbf{M}$ we work with are — unless explicitly stated or obvious from the context that this is not the case — assumed to be nonstandard countable models of $I\Delta_0 + \Omega_3$.

We shall consider various sequences of closed terms. About such a sequence Λ we shall always assume that if a term of the form $s^\varphi(\bar{t})$ appears in Λ, then all terms in $\bar{t}$ also do, and moreover, that they have smaller indices in Λ than $s^\varphi(\bar{t})$. Also, whenever dealing with a sequence Λ and a model $\mathbf{M}$, we shall assume that $lh(\Lambda)$ is in $log(\mathbf{M})$.

Given a sequence of terms Λ, let the collection $\mathcal{A}(\Lambda)$ of atomic sentences over Λ consist of all sentences obtained by substituting terms from Λ for variables in atomic formulae of L. Observe that there is a standard polynomial $\pi(n)$ such that $lh(\mathcal{A}(\Lambda)) \leq \pi(lh(\Lambda))$. Let us fix some such π.

Some more notation: if $\mathcal{F}$ is a class of formulae, the symbol $\mathcal{F}(\mathbf{M})$ denotes the family of all $\mathcal{F}$–definable elements of $\mathbf{M}$, while $\mathbf{M}^{\mathcal{F}}$ denotes the set of $\mathcal{F}$–sentences true in $\mathbf{M}$.

Finally, let us recall some relevant facts about universal formulae. Firstly, in $I\Delta_0 + exp$ there is a Σ_1 universal formula Sat for Δ_0. Thus, Sat is Σ_1, and for any $\mathbf{M} \models I\Delta_0 + exp$, $\varphi \in \mathbf{M}$ a Δ_0 formula,

$$\mathbf{M} \models Sat(\varphi) \text{ iff } \mathbf{M} \models \varphi.$$

Secondly, in $I\Delta_0 + \Omega_3$ there is an $\exists_r$ universal formula Sat_r for $\exists_r$, for each $r \in \omega$. Sat_r can obviously be also used as a universal formula for E_r, and additionally, if we limit our attention to the truth of E_r formulae smaller than some a with parameters smaller than some b, then the initial existential quantifier in Sat_r can also be bounded (thus giving an "E_r formula with a parameter": call this formula Sat_{E_r}).

2 Evaluations and evaluation models

Let $p : \mathcal{A}(\Lambda) \longrightarrow \{0,1\}$ map every axiom of equality in $\mathcal{A}(\Lambda)$ to 1. We call such a p an *evaluation* on Λ, since we may think of p as assigning a logical value to sentences in $\mathcal{A}(\Lambda)$ (see also [A1], [A2], [A3], [AZ1], [AZ2], [S]). Of course, p can be uniquely extended to all boolean combinations of sentences in $\mathcal{A}(\Lambda)$ in the routine way.

Note in passing that any evaluation on Λ is an object of size at most $exp(lh(\mathcal{A}(\Lambda)))$ and thus at most $exp(\pi(lh(\Lambda)))$.

For $\varphi(\bar{x})$ in negation normal form, $\bar{t} \in \Lambda$, we define the notion that Λ is *good enough* (g.e.) for $\langle \varphi, \bar{t} \rangle$ by induction on φ. Λ is always g.e. for $\langle \varphi, \bar{t} \rangle$ if φ is open. Λ is g.e. for $\langle \varphi_1 \vee \varphi_2, \bar{t} \rangle$ iff it is g.e. for $\langle \varphi_1, \bar{t} \rangle$ and $\langle \varphi_2, \bar{t} \rangle$, similarly for conjunctions. If φ is $\exists y \varphi'(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^{\varphi}(\bar{t}) \in \Lambda$ and Λ is g.e. for $\langle \varphi', \bar{t}^\frown s^{\varphi}(\bar{t}) \rangle$. Finally, if φ is $\forall y \tilde{\varphi}(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^{\exists y \neg \tilde{\varphi}(\bar{x},y)}(\bar{t}) \in \Lambda$ (where $\exists y \neg \tilde{\varphi}$ is the normal form of $\neg \varphi$) and Λ is g.e. for $\langle \tilde{\varphi}, \bar{t}^\frown s^{\exists y \neg \tilde{\varphi}(\bar{x},y)}(\bar{t}) \rangle$.

The idea is that Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if it contains enough appropriate Skolem terms so that assigning a logical value to $\varphi(\bar{t})$ based on an evaluation on Λ makes sense.

Definition 2.1 *Let $\bar{t} \in \Lambda$. We define the relation $p \models \varphi(\bar{t})$ for $\varphi(\bar{x})$ in negation normal form by induction:*

(i) $p \models \varphi(\bar{t})$ iff $p(\varphi(\bar{t})) = 1$ for $\varphi(\bar{t})$ open;

(ii) the relation $p \models \varphi$ behaves in the natural way with respect to conjunctions and disjunctions;

(iii) if φ is $\exists y \varphi'(\bar{x}, y)$, then $p \models \varphi(\bar{t})$ iff Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ and $p \models \varphi'(\bar{t}, s^{\varphi}(\bar{t}))$,

(iv) if φ is $\forall y \tilde{\varphi}(\bar{x}, y)$, then $p \models \varphi(\bar{t})$ iff for all $t \in \Lambda$ such that Λ is g.e. for $\langle \tilde{\varphi}, \bar{t}^\frown t \rangle$, $p \models \tilde{\varphi}(\bar{t}, t)$.

We will be especially interested in the case where Λ is one of a number of canonical sequences of terms. To define these, let $K(i)$ be the unique function satisfying $K(0) = 1$ and $K(i+1) = c \cdot exp(i) \cdot K(i)^i$, where c is an appropriately large standard integer. Note that for any i, $K(i) \in log$, as for almost all i, $K(i) \leq exp(i^i)$, and i^i is always in $log^{(2)}$, since we have:

Remark 2.2 *In any model of $I\Delta_0 + \Omega_3$, log is closed under ω_2, $log^{(2)}$ is closed under ω_1, $log^{(3)}$ is closed under multiplication, and $log^{(4)}$ is closed under addition.*

The notion of *canonical sequence of rank* i, Λ_i, is now defined by induction. Λ_{i+1} is the smallest sequence Λ such that:

- for any $j \leq i+1$, Λ contains the term s^j and is good for $\langle exp^{(3)}(x) = y, \underline{j}^\frown s_j \rangle$;
- for any $a \leq K(i)$, Λ contains the numeral $\underline{a}$, and if $exp(a) \leq exp^{(3)}(i+1)$, then Λ is g.e. for $\langle \exists exp(x), \underline{a} \rangle$;
- for any formula $\varphi < exp(i)$ of the form ψ^t or $\exists x \leq t\, \psi^x$ (where $t \in \Lambda_i$), and any $\bar{t} \in \Lambda_i$, Λ is g.e. for $\langle \varphi, \bar{t} \rangle$.

Observe that if c is chosen large enough, then $lh(\Lambda_i) \leq K(i)$ for all i (since a formula smaller than $exp(i)$ contains at most i quantifiers).

Some particularly well–behaved evaluations on Λ_i will be called evaluations *of rank* i (we let $\mathcal{A}_i$ stand for $\mathcal{A}(\Lambda_i)$):

Definition 2.3 *A function* $p : \mathcal{A}_i \longrightarrow \{0,1\}$ *is called an evaluation of rank* i *if the following holds:*

(i) for every $\varphi(\bar{x}) < exp(i)$ *and every* $\bar{t} \in \Lambda_i$ *of appropriate length, if* Λ_i *is g.e. for* $\langle \varphi, \bar{t} \rangle$*, then for all* $j \leq i$*,*

$$p \models \varphi(\bar{t})^{s_j} \textit{ or } p \models \neg\varphi(\bar{t})^{s_j};$$

(ii) if $\varphi < exp(i)$ *is an axiom of* $I\Delta_0^*$*, then assuming* Λ_i *is g.e. for* $\langle \varphi, \emptyset \rangle$*,* $p \models \varphi$*;*

(iii) $p \models (\underline{0} = 0 \wedge \underline{1} = 1)$*, and given any* $\underline{a}, \underline{b} \in \Lambda_i$*: if* $\underline{a+b} \in \Lambda_i$*, then* $p \models (\underline{a} + \underline{b} = \underline{a+b})$*, and similarly for the other symbols of* L*;*

(iv) for all $\underline{a} \in \Lambda_i$ *such that* $exp(a) \leq exp^{(3)}(i+1)$*,* $p \models \exists exp(\underline{a})$*;*

(v) for all $j \leq i$*,* $p \models s_j = exp^{(3)}(j)$*.*

We let "$p \in \mathcal{E}_i$" stand for "p is an evaluation of rank i". This is a slight abuse of notation, since the code for the set of evaluations of rank i might be too large to be an element of the model.

We claim that both "$p \models \varphi$" (for p an evaluation on Λ_i) and "$p \in \mathcal{E}_i$" are Δ_0 definable with an appropriately large parameter (and thus Δ_1 definable).

To see whether an evaluation p on Λ_i sets φ to "True" (i.e. whether $p \models \varphi$), we need to deal with sets $V_0, \ldots, V_r$, where where V_l is the set of values given by p to the l–th subformula of φ under all relevant substitutions of terms in Λ_i for the free variables in that subformula. Since there are at most $log\,\varphi$ variables in any subformula of φ, the number of possible substitutions is not greater than $K(i)^{log\,\varphi}$, and hence $V_l \leq exp(K(i)^{log\,\varphi})$. Again, there can be no more than $log\,\varphi$ subformulae of φ. Thus, the sequence $\langle V_0, \ldots, V_r\rangle$ is at most $log\,\varphi$–long, so its code is at most $exp(log\,\varphi \cdot K(i)^{log\,\varphi})$. This is the largest object relevant to the truth value given to φ by ψ, which shows that "$p \models \varphi$" is indeed Δ_0 definable with a parameter.

To see whether an evaluation p on Λ_i is in $\mathcal{E}_i$, we have to check what truth value it assigns to a number of formulae φ smaller than $exp(i)$. With some additional work, one may verify that all objects we need to consider are smaller than $\omega_3(exp^{(3)}i)$, which implies that also "$p \in \mathcal{E}_i$" is Δ_0 definable with a parameter.

We let $True(p, i, \varphi)$ be a Δ_1 formula which says "$p \in \mathcal{E}_i$ and $p \models \varphi$".

Definition 2.4 *If $p_1 \in \mathcal{E}_{i_1}$ and $p_2 \in \mathcal{E}_{i_2}$ with $i_1 \leq i_2$, we say that p_2 extends p_1 iff $p_1 \subseteq p_2$.*

The following proposition lists some "conservativity" relationships between evaluations one of which extends the other. The proofs are simple inductive arguments.

Proposition 2.5 *Let $p_2 \in \mathcal{E}_{i_2}$ extend $p_1 \in \mathcal{E}_{i_1}$. Then:*

(1) if Λ_{i_1} is g.e. for $\langle\varphi, \bar{t}\rangle$ and $j \leq i_1$, then $p_1 \models \varphi(\bar{t})^{s_j}$ iff $p_2 \models \varphi(\bar{t})^{s_j}$;

(2) if $\varphi(\bar{x})$ is an open formula and $\bar{t} \in \Lambda_{i_1}$, then $p_1 \models \varphi(\bar{t})$ iff $p_2 \models \varphi(\bar{t})$;

(3) if Λ_{i_1} is g.e. for $\langle\varphi, \bar{t}\rangle$, then $p_2 \models \varphi(\bar{t})$ implies $p_1 \models \varphi(\bar{t})$.

Another simple fact about evaluations is:

Proposition 2.6 *Let $p \in \mathcal{E}_i$ and let $i' \leq i$. Then $(p \restriction \mathcal{A}_{i'}) \in \mathcal{E}_{i'}$.*

The importance of evaluations consists in the fact that they make possible the construction of models for $I\Delta_0 + \Omega_3$. More precisely, such a model is determined by an ascending chain of evaluations whose ranks are cofinal in $log^{(3)}$ (note that by remark 2.2, in a model of $I\Delta_0 + \Omega_3$ the third logarithm has no last element).

Let $P = \langle p_n : n \in \omega\rangle$ be such a chain. If $\bar{t}$ is a tuple of terms of L_T and $\varphi(\bar{t})$ is open, then almost all p_n's agree on the logical value of $\varphi(\bar{t})$. So, we may define $P \models \varphi(\bar{t})$ by:

$$p_n \models \varphi(\bar{t}) \text{ for almost all } n.$$

We define the relation $=_P$ between terms in L_T by:

$$t_1 =_P t_2 \text{ iff } P \models (t_1 = t_2).$$

Since the p_n's are evaluations, $=_P$ is an equivalence relation and a congruence with respect to the relations of L. Let $\mathbf{M}_0[P]$ be the model whose universe is the set of $=_P$–equivalence classes and whose relations are defined by

$$[t_1] + [t_2] = [t_3] \text{ iff } P \models (t_1 + t_2 = t_3) \text{ etc.}$$

Clearly, we have:

$$\mathbf{M}_0[P] \models \varphi([\bar{t}]) \text{ iff } P \models \varphi(\bar{t})$$

for any open φ. If we introduce the more general relation $P \models \varphi(\bar{t})$, for φ not necessarily open, by the same clause as above, then induction yields

$$P \models \varphi(\bar{t}) \text{ implies } \mathbf{M}_0[P] \models \varphi([\bar{t}]).$$

The converse implication will not generally hold unless we accept a more restrictive definition of evaluation which is not needed here.

The next lemma and corollary show that the numeral $\underline{a}$ may be treated as a name for the a–th element of $\mathbf{M}_0[P]$.

Lemma 2.7 *Let $p \in \mathcal{E}_i$. If for a term $t \in L_T$, $p \models (t \leq \underline{a})$, then there is $b \leq a$ such that $p \models (t = \underline{b})$.*

Moreover, if φ is an open formula and $\underline{\bar{a}}$ is a tuple of numerals for numbers less or equal to $K(i-1)$, then $\varphi(\bar{a})$ implies $p \models \varphi(\underline{\bar{a}})$.

Proof. We may assume that our formalization of $I\Delta_0^*$ contains axioms such as: $\forall x(x \leq 0 \Rightarrow x = 0)$, $\forall x \forall y(x \leq y+1 \Rightarrow x = y+1 \vee x \leq y)$, $\forall x(x+0 = x)$, $\forall x \forall y((x+(y+1) = (x+y)+1)$ and similar axioms for the other symbols of L.

The first part of the lemma is proved by induction on $a \leq K(i-1)$. For $a = 0$, $p \models (\underline{0} = 0)$, so $p \models (t \leq \underline{0})$ implies $p \models (t \leq 0)$, hence $p \models (t = 0)$ by the appropriate axiom, hence $p \models (t = \underline{0})$. Assume that the thesis holds for a and that $p \models (t \leq \underline{a+1})$. Then, since $p \models (\underline{a+1} = \underline{a} + 1)$, we get either $p \models (t = \underline{a+1})$ or $p \models (t \leq \underline{a})$, in which case we use the inductive assumption to get $p \models (t = \underline{b})$ for some $b \leq a$.

For the "moreover" part, first prove $p \models (\underline{a_1} + \underline{a_2} = \underline{a_1 + a_2})$ (assuming $a_1 + a_2 \leq K(i-1)$) by induction, using appropriate axioms for the induction base and induction step. Then proceed similarly with

$p \models (\underline{a_1} \cdot \underline{a_2} = \underline{a_1 \cdot a_2})$ (again, assuming i is large enough) and the remaining symbols of L, and pass through boolean combinations to obtain the thesis. $\square$

Corollary 2.8 *The mapping $a \to [\underline{a}]$ (for $a \in log(\mathbf{M})$ is an isomorphism between $log(\mathbf{M})$ and an initial segment I of $\mathbf{M}_0[P]$*

Proof. It suffices to observe that if $P = \langle P_n : n \in \omega \rangle$ where $p_n \in \mathcal{E}_{i_n}$, then for any tuple $\bar{a} \in log$, the maximal element of $\bar{a}$ is smaller than $K(i_n - 1)$ for almost all n, so we may apply lemma 2.7. $\square$

By clause (iv) of definition 2.3, $I \subseteq log(\mathbf{M}_0[P])$. Let $\mathbf{M}[P]$ be the initial segment of $\mathbf{M}_0[P]$ generated by $exp(I)$. If we identify I with $log(\mathbf{M})$, we obtain:

Corollary 2.9 *$log(\mathbf{M}) = log(\mathbf{M}[P])$. Thus, more generally, $log^{(n)}(\mathbf{M}) = log^{(n)}(\mathbf{M}[P])$ for all $n \geq 1$.*

We also have:

Corollary 2.10 *If $\varphi(\bar{x})$ is a Π_1 formula and $P \models \varphi(\bar{t})$, then $\mathbf{M}[P] \models \varphi([\bar{t}])$*

We close this section with a theorem on evaluation models (i.e. models of the form $\mathbf{M}[P]$) which will play a key role later on.

Theorem 2.11 *Let $\mathbf{M}$ be a countable model of $I\Delta_0 + \Omega_3 + B\Sigma_1$. Assume that $\mathcal{F}$ is a set of standard L–formulae,*

$$\mathcal{F} = \{\theta_n(x_1, \ldots, x_r) : n \in \omega\},$$

and is a subset of a set

$$\{\theta_l(x_1, \ldots, x_r) : l \in \log^{(3+k)} \mathbf{M}\}$$

(for some $k \in \omega$) which is Δ_1–definable in $\mathbf{M}$ and satisfies

$$(\#)\ \forall i\ \exists p \in \mathcal{E}_i\ \forall l, l_1, \ldots, l_r < \log^{(k)} i\ \ p \models \theta_l(\underline{l_1}, \ldots, \underline{l_r}).$$

Then there exists an increasing and cofinal sequence $P = \langle p_n : n \in \omega \rangle$ of evaluations such that $P \models \varphi(\underline{l_1}, \ldots, \underline{l_r})$ for each $\varphi \in \mathcal{F}$, $\underline{l_1}, \ldots, \underline{l_r} \in log^{(3+k)}(\mathbf{M})$, and the model $\mathbf{M}[P]$ satisfies $I\Delta_0 + \Omega_3$.

In particular, for any n such that θ_n is Π_1, $\mathbf{M}[P] \models \theta_n(l_1, \ldots, l_r)$, for each $l_1, \ldots, l_r \in \log^{(3+k)}(\mathbf{M})$, $n \in \omega$.

Proof. Let us introduce the following convention: every evaluation p of rank i appearing in this proof satisfies $p \models \theta_l(\underline{l_1}, \ldots, \underline{l_r})$ for all $l, l_1, \ldots, l_r < \log^{(k)} i$.

Let $i_1 < i_2 < \ldots$ be cofinal in $\log^{(3)}(\mathbf{M})$. We shall define a sequence $P = \langle p_n : n \in \omega \rangle$ such that $p_n \in \mathcal{E}_{i_n}$.

P is defined by induction as follows. Suppose that at a given step n we already have evaluations $p_1 \subseteq \ldots \subseteq p_n$ such that $p_1 \in \mathcal{E}_{i_1}, \ldots, p_n \in \mathcal{E}_{i_n}$ satisfying the inductive condition

$$(*)\ \forall i > i_n\ \exists p \in \mathcal{E}_i \big[p_n \subseteq p\big].$$

Note that at the initial step the validity of the inductive condition is ensured by the assumption of the theorem.

We claim that it follows by $B\Sigma_1$ that:

$$(**)\ \exists p_{n+1} \in \mathcal{E}_{i_{n+1}} \big[\forall i > i_{n+1}\ \exists p \in \mathcal{E}_i\ p_n \subseteq p_{n+1} \subseteq p\big].$$

Indeed, assume $(**)$ fails. Then for any $\tilde{p} \in \mathcal{E}_{i_{n+1}}$ extending p_n there exists $i(\tilde{p}) > i_{n+1}$ for which there is no evaluation $p \in \mathcal{E}_i$ extending $\tilde{p}$. Now, all $\tilde{p}$'s are bounded by $exp(\pi(K(i_{n+1}))$. Thus, we may use $B\Sigma_1$ to find a common bound i for all the $i(\tilde{p})$'s. It follows that there is no $p \in \mathcal{E}_i$ extending any of the $i(\tilde{p})$'s. On the other hand, by $(*)$ there is some $p \in \mathcal{E}_i$ extending p_n. But $(p \restriction \mathcal{A}_{i_{n+1}}) \in \mathcal{E}_{i_{n+1}}$, and $p_n \subseteq (p \restriction \mathcal{A}_{i_{n+1}}) \subseteq p$, a contradiction. Hence, $(**)$ must hold and the claim is proved.

Finally, the evaluation p_{n+1} given by $(**)$ satisfies the inductive condition at stage $n+1$.

Now let $P = \langle p_n : n \in \omega \rangle$. Obviously P is increasing and cofinal.

Since all the axioms of $I\Delta_0^*$ are Π_1 we infer from corollary 2.10 that

$$\mathbf{M}[P] \models I\Delta_0^*.$$

On the other hand, the set $\{exp^{(3)} i : i \in \log^{(3)} \mathbf{M}\}$ is cofinal in both $\mathbf{M}$ and $\mathbf{M}[P]$ (cf. corollary 2.9). Since $\mathbf{M} \models \Omega_3$, we infer in view of corollary 2.9 that $\mathbf{M}[P] \models \Omega_3$. Consequently, $\mathbf{M}[P] \models I\Delta_0 + \Omega_3$ since obviously $I\Delta_0^* + \Omega_3$ implies $I\Delta_0 + \Omega_3$. This completes the proof of the theorem. $\square$

Remark 2.12 *To keep the enunciation of the above theorem reasonably concise, we have formulated its assumptions in a relatively simple way. It is clear, however, that appropriate variants of the theorem would also be true if the assumptions were modified in one or more of the following ways:*

- *in (#),* $\forall i \exists p \in \mathcal{E}_i(\ldots)$ *could be replaced by* $\forall^\infty i \exists p \in \mathcal{E}_i(\ldots)$*;*

- *also in (#),* $\forall l, l_1, \ldots, l_r < log^{(k)}i$ *could be replaced by* $\forall l, l_1, \ldots, l_r < (log^{(k)}i)/r$ *(for any standard* r*), as long as* $log^{(3+k)}$ *is closed under addition;*

- $\mathcal{F}$ *could be extended by adding finitely many formulae of the form* $\varphi(\bar{t})$ *evaluated to "True" by almost all of the* p*'s given by (#).*

In the sequel, we will sometimes speak of using "theorem 2.11" when some such variant is actually meant.

3 The principle τ

The present section introduces a consistent sentence τ which is a kind of reflection principle (mentioned in the title). We begin by formulating some preservation properties of evaluations.

For a Σ_1 sentence Φ of the form $\exists x \phi^x$ let $\Gamma_\Phi(p,i)$ be the formula

$$\forall j \leq i\big(\exists x \leq exp^{(3)}j\ \phi^x \Rightarrow True(p,i,\exists x \leq s_j \phi^x)\big)$$
$$\wedge\ \big(\forall x \leq exp^{(3)}i\ \neg\phi^x \Rightarrow True(p,i,\forall x \neg\phi^x\big),$$

and, for a fixed sufficiently large m which depends on some further constructions but could be specified in advance, let $\Gamma_m(p,i)$ be the formula

$$\forall \psi < i, \psi \in \exists_m\ \forall j \leq i\ \forall \underline{a_1}, \ldots, \underline{a_r} \in \Lambda_i$$
$$\big(Sat_m(\psi^{exp^{(3)}j}(a_1,\ldots,a_r)) \Rightarrow True(p,i,\psi^{s_j}(\underline{a}_1,\ldots,\underline{a}_r)\big).$$

Intuitively, $\Gamma_\Phi(p,i)$ says "p preserves the size of a witness for $\Phi = \exists x \phi^x$, and disallows witnesses of size greater than $exp^{(3)}i$", while $\Gamma_m(p,i)$ says "p preserves the restrictions $\psi^{exp^{(3)}j}$, for $j \leq i$, of all $\exists_m$ sentences smaller than i".

Arguments similar to those in the previous section show that both Γ_Φ and Γ_m are Δ_0 with a parameter (and hence Δ_1), as they make no reference to objects greater than $\omega_3(exp^{(3)}i)$.

We will now define some (possibly non–standard) sentences τ_{j,j_1} for $j, j_1 \in log^{(4)}$. The definition is by induction on j_1. Let $\tau_{j,0}$ be:

$$\Big(\exists p \in \mathcal{E}_{exp\,0}\{\Gamma_m(p, exp\,0) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp\,0)\}\Big)^{exp^{(4)}2\cdot\underline{0}},$$

and let τ_{j,j_1+1} be:

$$\Big(\exists p \in \mathcal{E}_{exp(\underline{j_1+1})}\{\Gamma_m(p, exp(\underline{j_1+1})) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp(\underline{j_1+1})) \\ \wedge \bigwedge_{l,l_1<((j_1+1)/2)} True(p, exp(\underline{j_1+1}), \underline{\tau_{l,l_1}})\}\Big)^{exp^{(4)}2(\underline{j_1+1})}.$$

If the definition of τ_{j,j_1} is to make sense, an evaluation of rank $exp\ j_1$ should be able to decide the truth value of τ_{l,l_1} for $l, l_1 < j_1/2$. To check that this is so, let $\varphi_{j,j_1}(z, \bar{x})$ stand for

$$\Big(\exists y \in \mathcal{E}_{exp\ z}\{\Gamma_m(y, exp\ z) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(y, exp\ z) \\ \wedge \bigwedge_{l,l_1<(j_1/2)} True(y, exp(z), x_{l,l_1})\}\Big)^{exp^{(4)}(2z)},$$

and $\bar{t}_{j_1}$ stand for $\langle \underline{\tau_{l,l_1}} : l, l_1 < (j_1/2)\rangle$.

Observe that for τ_{j,j_1} is $\varphi_{j,j_1}(\underline{j_1}, \bar{t}_{j_1})$. Therefore, it is enough to check that for any j_1, $\varphi_{j_1/2,j_1/2}$ is smaller than $exp(exp(j_1)-1)$ and that $\bar{t}_{j_1/2} \in \Lambda_{exp(j_1)-1}$.

To see the former, note that given any j_1 a code for φ_{j_1,j_1} is about $j_1^{j_1^2}$, which is smaller than $\omega_1(exp\ j_1)$ (a precise bound on φ_{j_1,j_1} depends on the details of how we code the syntax, esp. the variables, but the main ingredient of φ_{j_1,j_1} is a $(j_1^2/4)$–long conjunction of formulae whose codes will not greatly exceed the code for the $(j_1^2/4)$-th variable, which in turn may be around j_1^2). So for us it suffices if $\omega_1(exp(j_1/2))$ is smaller than $exp(exp(j_1) - 1)$, which is clearly always the case.

To see the latter, we only need to check that for all j_1, $\tau_{j_1/4,j_1/4}$ is smaller than $K(exp(j_1) - 1)$. But for any j_1, the size of τ_{j_1,j_1} can be bounded by roughly $\omega_1(exp\ j_1)$ (the code for φ_{j_1,j_1}) times the code for the $(j_1^2/4)$–long sequence of the $\underline{\tau_{l,l_1}}$'s (for $l, l_1 < j_1/2$). This sequence will have a code smaller than $(3 \cdot \tau_{j_1/2,j_1/2})^{j_1^2/4}$. Using the fact that $K(i+1) > cK(i)^i$ for some large standard c, it is easy to verify that $K(exp(j_1) - 1)$ is more than τ_{j_1,j_1} (not to mention $\tau_{j_1/4,j_1/4}$).

In addition to the τ_{j,j_1}'s we also define, for any $j \in log^{(4)}$, a formula $\tau_j(j_1)$ with j_1 as a free variable. $\tau_j(j_1)$ is:

$$\Big(\exists p \in \mathcal{E}_{exp\ j_1}\{\Gamma_m(p, exp\ j_1) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp\ j_1) \\ \wedge \forall l, l_1 < (j_1/2)\forall x\big(x = \tau_{l,l_1} \Rightarrow True(p, exp\ j_1, x)\big)\}\Big)^{exp^{(4)}(2j_1)},$$

where $x = \tau_{l,l_1}$ is an abbreviation for the inductive definition of τ_{l,l_1} with l and l_1 as parameters. Note that although the $\tau_j(\cdot)$'s are in general again non–standard, $\tau_n(\cdot)$ is a standard formula for any standard n.

Note also that $exp^{(4)}(2j_1)$ is not less than $\omega_3(exp^{(4)} j_1)$ — the greatest element we possibly need to access in order to check whether a given $p \in \mathcal{E}_{exp\, j_1}$ satisfies all the conditions required in τ_{j,j_1} or $\tau_j(j_1)$ (as long as j is not unreasonably large in comparison to j_1). For this reason, the relativization to $exp^{(4)}(2j_1)$, which is necessary for technical reasons, does not essentially influence the sense of τ_{j,j_1} or $\tau_j(j_1)$.

Let $\psi_j(z)$ stand for

$$\Big(\exists y \in \mathcal{E}_{exp\, z}\{\Gamma_m(y, exp\, z)) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(y, exp\, z)) \\ \wedge \forall l, l_1 < (z/2) \forall x \big(x = \tau_{l,l_1} \Rightarrow True(y, exp\, z, x)\big)\}\Big)^{exp^{(4)}(2z)},$$

where $x = \tau_{l,l_1}$ is an abbreviation for the inductive definition of τ_{l,l_1}.

The following lemma establishes an important connection between τ_{j,j_1} and $\tau_j(j_1)$.

Lemma 3.1 *Let $j \leq j_1$ and let i be such that:*

- *The formulae $x = \tau_{l,l_1}$ (as a formula of x, l, l_1) and $True(y, exp\, z, x)$ may be bounded by $exp^{(3)}(i/2)$ for any choice of $l, l_1 < j_1/2$, $z < j_1$, $y < exp(\pi(K(exp\, j_1)))$, and $x < K(exp\, j_1)^{exp\, j_1}$;*
- *Λ_i is g.e. for $\langle \varphi_{j,j_1}, \underline{j_1}^\frown \bar{t}_{j_1} \rangle$ and for $\langle \psi_{j,j_1}, \underline{j_1} \rangle$.*

Let $p \in \mathcal{E}_i$ satisfy $\Gamma_m(p, i)$.
Then $\mathbf{M} \models True(p, i, \tau_{j,j_1})$ iff $\mathbf{M} \models True(p, i, \tau_j(\underline{j_1}))$

Remark 3.2 *Any $i \geq exp(2j_1)$ satisfies the conditions of the lemma.*

Proof. We prove the left-to-right direction as the other direction is very similar.

Assume $\mathbf{M} \models True(p, i, \tau_{j,j_1})$.

As already noted, τ_{j,j_1} is $\varphi_{j,j_1}(\underline{j_1}, \bar{t}_{j_1})$. So, by the definition of τ_{j,j_1} and the meaning of the formula $True$, it follows that for all $l, l_1 < j_1/2$,

$$p \models True(s^{\varphi_{j,j_1}}(\underline{j_1}^\frown \bar{t}_{j_1}), exp\, \underline{j_1}, \underline{\tau_{l,l_1}}).$$

We may assume that m was chosen large enough so that the formula $x = \tau_{l,l_1}$ is $\exists_m$. Then, by our assumptions on the size of i, we may use the fact that p satisfies Γ_m to get $p \models (\underline{\tau_{l,l_1}} = \tau_{\underline{l},\underline{l_1}})$ for all $l, l_1 < j_1/2$. Thus, for every t such that $p \models (t = \tau_{\underline{l},\underline{l_1}})$, we also have $p \models (t = \underline{\tau_{l,l_1}})$. By definition 2.1, this means that

$$p \models \forall x (x = \tau_{\underline{l},\underline{l_1}} \Rightarrow True(s^{\varphi_{j,j_1}}(\underline{j_1}^\frown \bar{t}_{j_1}), exp\, \underline{j_1}, x))$$

for any choice of $l, l_1 < j_1/2$.

Similarly, for every t such that $p \models (t < \underline{j_1}/2)$, we also have $p \models (t = \underline{l})$ for some $l < j_1/2$. Therefore, we get

$$p \models \forall l, l_1 < \underline{j_1}/2\, \forall x(x = \tau_{l,l_1} \Rightarrow True(s^{\varphi_{j,j_1}}(\underline{j_1}{}^\frown \bar{t}_{j_1}), exp\, \underline{j_1}, x)).$$

Combining this with the original assumption that $\mathbf{M} \models True(p, i, \tau_{j,j_1})$, we obtain:

$$p \models \Big(\{\Gamma_m(s^{\varphi_{j,j_1}}(\underline{j_1}{}^\frown \bar{t}_{j_1}), exp\, \underline{j_1}) \wedge \bigwedge_{\Phi \leq j} \Gamma_\Phi(s^{\varphi_{j,j_1}}(\underline{j_1}{}^\frown \bar{t}_{j_1}, exp\, \underline{j_1})$$
$$\wedge \forall l, l_1 < \underline{j_1}/2\, \forall x(x = \tau_{l,l_1} \Rightarrow True(s^{\varphi_{j,j_1}}(\underline{j_1}{}^\frown \bar{t}_j), exp(\underline{j_1}), x)\}\Big)^{exp^{(4)}(2\underline{j_1})}.$$

To prove $\mathbf{M} \models True(p, i, \tau_j(\underline{j_1}))$, we only need to check that p also evaluates the above formula to "True" if we substitute the appropriate Skolem term for $s^{\varphi_{j,j_1}}(\underline{j_1}{}^\frown \bar{t}_{j_1})$. If that was not the case, we would have neither $p \models \tau_j(\underline{j_1})$ nor $p \models \neg\tau_j(\underline{j_1})$ (since we have a witness for the initial existential quantifier in $\tau_j(\underline{j_1})$). But $p \models (s_{exp\, 2j_1} = exp^{(4)}(2\underline{j_1}))$, so p treats $\tau_j(\underline{j_1})$ as a formula relativized to $s_{exp\, 2j_1}$. Now, $p \in \mathcal{E}_i$, and thus it follows from part (i) of definition 2.3 that at least one of $p \models \tau_j(\underline{j_1})$ and $p \models \neg\tau_j(\underline{j_1})$ must hold. $\square$

Corollary 3.3 *Let $j \leq j_1$. Then*

$$\mathbf{M} \models \forall i\, (\Lambda_i \textit{ g.e. for } \langle \varphi_{j,j_1}, \underline{j_1}{}^\frown \bar{t}_{j_1} \rangle \Rightarrow \exists p \in \mathcal{E}_i\, True(p, i, \tau_{j,j_1}))$$

iff

$$\mathbf{M} \models \forall i\, (\Lambda_i \textit{ g.e. for } \langle \psi_j, \underline{j_1} \rangle \Rightarrow \exists p \in \mathcal{E}_i\, True(p, i, \tau_j(\underline{j_1})).$$

Proof. Follows from the lemma via propositions 2.5 and 2.6. $\square$

We now let τ be $\forall j \forall j_1 Sat(\tau_j(j_1))$.

In view of lemma 3.1, the sentence τ can be treated as a form of reflection principle (an observation due to A. Blass). Indeed, a "Π_1 reflection principle" is usually understood to be a formalized version of the principle

$$(*)\ \psi \text{ is provable} \Rightarrow \psi \text{ is true},$$

for $\psi \in \Pi_1$, in other words,

$$(**)\ \phi \text{ is true} \Rightarrow \phi \text{ is consistent},$$

for $\phi \in \Sigma_1$. Now, the existence of evaluations which satisfy ϕ is a kind of consistency of ϕ. So, in any model in which Sat is well–behaved as a truth definition, τ says:

$$\phi \text{ is true} \Rightarrow \phi \text{ plus a restricted fragment of } \tau \text{ is consistent},$$

for $\phi \in \Sigma_1$. Thus, τ expresses $(**)$ and additionally has a limited "self–reproducing" property.

As remarked above, τ is a consistent sentence. Even more:

Theorem 3.4 *The theory $I\Delta_0 + exp$ proves τ.*

Proof. Let us work in a model of $I\Delta_0 + exp$. We prove $\forall j \leq j_1 Sat(\tau_j(j_1))$ by induction on j_1.

Assume $\forall j \leq j_1 Sat(\tau_j(j_1))$. We want to show $\forall j \leq j_1 + 1\ Sat(\tau_j(j_1))$. Thus, given any $j \leq j_1 + 1$, we need

$$\Big(\exists p \in \mathcal{E}_{exp(j_1+1)}\{\Gamma_m(p, exp(j_1+1)) \wedge \textstyle\bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp(j_1+1)))$$
$$\wedge\, \forall l, l_1 < ((j_1+1)/2)\, \forall x\big(x = \tau_{l,l_1} \Rightarrow True(p, exp(j_1+1), x)\big)\}\Big)^{exp^{(4)}(2(j_1+1))}.$$

We will find an evaluation p of rank $j_1 + 1$ such that

$$(*)\ \Gamma_m(p, exp(j_1+1)) \wedge \textstyle\bigwedge_{\Phi \leq j} \Gamma_\Phi(p, exp(j_1+1))$$
$$\wedge\, \forall l, l_1 < ((j_1+1)/2)\, (True(p, exp(j_1+1), \tau_l(\underline{l_1})).$$

The fact that p is as required in $\tau_j(j_1+1)$ will then follow from lemma 3.1, since $exp(j_1+1)$ is a large enough rank for the lemma to ensure the equivalence of $p \models \tau_{l,l_1}$ and $p \models \tau_l(\underline{l_1})$ for $l \leq l_1 < (j_1+1)/2$ (see remark 3.2).

The way to obtain p is by constructing a *Skolem hull* on $\Lambda_{exp(j_1+1)}$. A Skolem hull on a given Λ is a sequence $H = \langle h_t : t \in \Lambda \rangle$ of elements of M, where the element h_t is thought of as an interpretation of the term t. One may define the satisfaction relation $H \models \varphi(\bar{t})$ in much the same way as $p \models \varphi(\bar{t})$, i.e. by postulating

- $H \models \varphi(\bar{t})$ iff $\mathbf{M} \models \varphi(\bar{h_t})$

for φ open, and then proceeding as in definition 2.1, so that e.g.

- $H \models \exists y \varphi'(\bar{t}, y)$ iff Λ_i is g.e. for $\langle \exists y \varphi'(\bar{x}, y), \bar{t} \rangle$ and $H \models \varphi'(\bar{h_t}, h_{s^\varphi(\bar{t})})$.

It is clear that any hull H on Λ determines an evaluation p_H such that $p_H \models \varphi$ iff $H \models \varphi$. If Λ is Λ_i, and H is a hull *of rank i* (defined analogously to "evaluation of rank i", cf. def. 2.3), then $p_H \in \mathcal{E}_i$.

The hull we want to construct on $\Lambda_{exp(j_1+1)}$ is to satisfy:

(i) for any $h \in H$, $h \leq exp^{(4)}(j_1+1)$,

(ii) for any $a \leq K(exp\, j_1)$, $h_{\underline{a}} = a$, and for any $j' \leq exp(j_1+1)$, $h_{s_{j'}} = exp^{(3)}\, j'$;

(iii) for any $h_t \in H$ and for any formula of the form ψ^t smaller than $exp(exp(j_1+1)-1)$,

$$(H \models \psi^t) \text{ iff } Sat(\psi^{h_t});$$

(iv) for any $h_t \in H$ and for any formula of the form $\exists x \leq t\, \psi^x$ smaller than $exp(exp(j_1+1)-1)$,

$$(H \models \exists x \leq t\, \psi^x) \text{ iff } Sat(\exists x \leq h_t \psi^x);$$

(v) for every $\varphi < exp(j_1+1)$,

$$\text{if } Sat(\forall x \leq exp^{(4)}(j_1+1) \neg\phi^x) \text{ then } H \models \forall x \neg\varphi^x.$$

The actual construction of H is based on a straightforward induction. Given that $\Lambda_{exp(j_1+1)}$ is ordered as $\langle t_1, \ldots, t_k\rangle$, we assign interpretations to the t_r's by induction on $r \leq k$. If t_{r+1} is $\underline{a}$, then $h_{t_{r+1}} = a$, if it is $s_{j'}$, then $h_{t_{r+1}} = exp^{(3)} j'$. If $t_{r+1} = s^{\varphi}(\bar{t})$, then we define $h_{t_{r+1}}$ to be the smallest witness below $exp^{(4)}(j_1+1)$ for $\varphi(\bar{h_t})$ whenever it exists, and arbitrary (but smaller than $exp^{(4)}(j_1+1)$) if there is no such witness.

We take p to be p_H. It is again straightforward to check that $p \in \mathcal{E}_{exp(j_1+1)}$ and that p has all the properties required in $(*)$. In particular, $\forall l, l_1 < ((j_1+1)/2)(True(p, exp(j_1+1), \tau_l(\underline{l_1}))$ follows by the construction of p from the inductive assumption $\forall j \leq j_1 Sat(\tau_j(j_1))$. $\square$

4 The main theorem

To define the theory T mentioned in the introduction, we will use "finite fragments" of the principle τ. Namely, let τ_n denote

$$\forall i\, \exists p \in \mathcal{E}_i \Big(\{ \Gamma_m(p,i)) \wedge \textstyle\bigwedge_{\Phi \leq n} \Gamma_\Phi(p,i)) \\ \wedge \forall l, l_1 < (log\, i)/2 \forall x \big(x = \tau_{l,l_1} \Rightarrow True(p,i,x) \big) \} \Big)^{(exp^{(3)}(2log\, i))}.$$

Thus, using the notation of the previous section, τ_n is (approximately) $\forall j_1 \tau_n(j_1)$. In particular, for $n \in \omega$, τ_n is a standard Π_1 sentence.

Lemma 4.1 *Any $\exists_m$ sentence χ consistent with $I\Delta_0 + \Omega_3$ is consistent with all the τ_n's.*

Proof. Let $\mathbf{M} \models I\Delta_0 + \Omega_3 + \tau_0 + \chi$. W.l.o.g. we may assume that $\mathbf{M} \models B\Sigma_1$, since (cf. e.g. [P]) $\mathbf{M}$ has a 1–elementary extension $\mathbf{M}'$ of the same height satisfying $B\Sigma_1$.

Let $\mathcal{F}$ be the set

$$\{\tau_n(x) : n \in \omega\} \cup \{\chi\}.$$

This is a subset of

$$\{\tau_l(x) : l \in \log^{(4)}\} \cup \{\chi\}.$$

Using the fact that $\mathbf{M} \models \tau_0$, we infer from (a minor variant of) corollary 3.3 that:

$$\forall i\, \exists p \in \mathcal{E}_i\, \forall l, j_1 < (log\, i)/2 \;\; p \models \tau_l(\underline{j_1}),$$

since Λ_i is g.e. for $\langle \psi_l, \underline{j_1} \rangle$ whenever $l, j_1 < (log\, i)/2$. Also, almost all the p's evaluate χ to "True", because all p's given by τ_0 satisfy Γ_m, and there is an i such that a witness for χ exists below $exp^{(3)} i$. Since $log^{(4)}$ is closed under addition, we may apply theorem 2.11 and obtain an increasing and cofinal sequence P_0 of evaluations such that: $\mathbf{M}[P_0] \models I\Delta_0 + \Omega_3$, $P_0 \models \chi$, and $P_0 \models \tau_n(\underline{l_1})$ for any $n \in \omega, j_1 \in log^{(4)}(\mathbf{M})$. Since the $\tau_n(\cdot)$'s are Π_1, it holds that $\mathbf{M}[P_0] \models \tau_n(j_1)$ for any n and j_1.

But this means that for any n, j_1,

$$\mathbf{M}[P_0] \models \exists p \in \mathcal{E}_{exp\, j_1} \{ \Gamma_m(p, exp\, j_1)) \wedge \textstyle\bigwedge_{\Phi \leq n} \Gamma_\Phi(p, exp\, j_1)) \\ \wedge \forall l, l_1 < j_1/2\, \forall x \big(x = \tau_{l,l_1} \Rightarrow True(p, exp\, j_1, x) \big) \}.$$

We may obtain suitable p's in $\mathcal{E}_i$ for i not of the form $exp\, j_1$ by restricting the evaluations we have in $\mathcal{E}_{exp\, j_1}$ (use propositions 2.6 and 2.5 to ensure that these restrictions are indeed evaluations of the appropriate ranks and that they have the desired properties). Hence, for any n it holds in $\mathbf{M}[P_0]$ that

$$\forall i \, \exists p \in \mathcal{E}_i \{\Gamma_m(p,i)) \wedge \textstyle\bigwedge_{\Phi \leq n} \Gamma_\Phi(p,i))$$
$$\wedge \, \forall l, l_1 < (log\, i)/2 \, \forall x\big(x = \tau_{l,l_1} \Rightarrow True(p,i,x)\big)\},$$

which implies that $\mathbf{M} \models \tau_n$ for all n. It remains to point out that $\mathbf{M}[P_0]$ also satisfies χ, since $P_0 \models \chi$ and we may in this context treat χ as a Δ_0 formula by considering its relativization to the smallest witness for χ. $\Box$

Observe that the construction described in the proof of the lemma would have also worked if we started in a model of some higher τ_N, and not just τ_0. In that case, we would be able to replace the set $\mathcal{F}$ by a set which additionally contains $\neg\Phi$, for all $\Phi \leq N$ false in $\mathbf{M}$, and suitable true relativizations of Φ, for $\Phi \leq N$ true in $\mathbf{M}$. Theorem 2.11 would then give us a sequence P_N corresponding to that set. $\mathbf{M}[P_N]$ would satisfy all the τ_n's and χ just as $\mathbf{M}[P_0]$ did, but it would additionally satisfy exactly those $\Phi \leq N$ which are true in $\mathbf{M}$.

Given any Π_1 consequence θ of $I\Delta_0 + exp$, there exists a purely existential sentence φ_θ consistent with $I\Delta_0 + \Omega_3 + B\Sigma_1 + \theta$, but inconsistent with $I\Delta_0 + exp$ (see [HP]). If we apply this result to $\theta := \tau_0$, we obtain an $\exists_m$ sentence χ consistent with $I\Delta_0 + \Omega_3 + B\Sigma_1 + \{\tau_n : n \in \omega\}$, but inconsistent with $I\Delta_0 + exp$. Let us fix such a χ and define the theory T_0 by

$$T_0 := I\Delta_0 + \Omega_3 + B\Sigma_1 + \{\tau_n : n \in \omega\} + \chi.$$

In the sequel ϵ will denote a binary sequence, $\epsilon = \langle \epsilon_0, \ldots, \epsilon_{lh(\epsilon)-1} \rangle$.

Given a fixed Δ_0 enumeration $\langle \varphi_n : n \in \omega \rangle$ of all Σ_1 sentences, let us introduce the sentences $\sigma_{n,\epsilon}$:

$$\sigma_{n,\epsilon} := (n = lh(\epsilon))$$
$$\wedge \, \forall i \, \exists p \in \mathcal{E}_i \Big(\Gamma_m(p,i) \wedge \forall l, l_1 < (log\, i)/2 \, \forall x (x = \tau_{l,l_1}) \Rightarrow True(p,i,x))$$
$$\wedge \, \forall k < n \, (\epsilon_k = 0 \Rightarrow p \models \neg\varphi_k) \wedge p \models \neg\varphi_n\Big).$$

Lemma 4.2 *Assuming m is sufficiently large, for any n and ϵ, $\sigma_{n,\epsilon}$ is (equivalent in $I\Delta_0 + \Omega_3$ to) a $\forall_m$ sentence.*

Proof. The only difficulty is to show that $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p,i)$ can be equivalently written as a $\forall_m$ formula. $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p,i)$ is:

$$\forall i \, \exists p \in \mathcal{E}_i \, \forall j \leq i \, \forall \psi < i, \psi \in \exists_m \, \forall \underline{a_1}, \ldots, \underline{a_r} \in \Lambda_i$$
$$\big(Sat_m(\psi^{exp^{(3)}j}(a_1, \ldots, a_r)) \Rightarrow True(p, i, \psi^{s_j}(\underline{a}_1, \ldots, \underline{a}_r))\big),$$

so the main problem is that an implication with the $\exists_m$ precedent Sat_m occurs in the scope of the bounded existential quantifier $\exists p$. Actually, we could clearly replace Sat_m by its E_m analogue Sat_{E_m} (see the Preliminaries), but this still does not solve our problem.

By the definition of $\exists_m$, an $\exists_m$ formula $\psi(a_1, \ldots, a_r)$ is $\exists x \psi'(a_1, \ldots, a_r, x)$ for $\psi' \in U_{m-1}$. Thus $\psi^{exp^{(3)}j}(a_1, \ldots, a_r)$ holds iff there is a witness $x \leq exp^{(3)}j$ such that $\psi'(a_1, \ldots, a_r, x)$.

Consider the sentence ξ_m

$$\begin{gathered}\forall i \, \forall \langle x_{j,\psi,\underline{\bar{a}}} : j \leq i, \psi < i \text{ in } \exists_m, \underline{\bar{a}} \in \Lambda_i \text{ of appropriate length} \rangle \\ \text{such that each } x_{j,\psi,\underline{\bar{a}}} \text{ is } \leq exp^{(3)}j \\ \exists p \in \mathcal{E}_i \, \forall j \leq i \, \forall \psi < i, \psi \in \exists_m \, \forall \underline{a_1}, \ldots, \underline{a_r} \in \Lambda_i \\ \big(Sat_{U_{m-1}}(\psi'(a_1, \ldots, a_r, x_{j,\psi,\underline{\bar{a}}})) \Rightarrow True(p, i, \psi^{s_j}(\underline{a}_1, \ldots, \underline{a}_r)\big),\end{gathered}$$

where $\langle x_{j,\psi,\underline{\bar{a}}} \rangle$ should be thought of as a sequence of "potential witnesses" smaller than $exp^{(3)}j$ for $\psi(\underline{\bar{a}})$, and $Sat_{U_{m-1}}$ is dual to $Sat_{E_{m-1}}$.

ξ_m is easily seen to be equivalent to a $\forall_m$ sentence. Indeed: $Sat_{U_{m-1}}$ is U_{m-1} with an appropriately large parameter, so it is E_{m-1} in the precedent of an implication; the universal quantifiers for j, ψ, and $\underline{\bar{a}}$ may be treated as sharply bounded (in particular, $\bar{a}$ is an at most $log\, \psi$–long sequence of objects smaller than $3K(i-1)$, so it is $\leq (3K(i-1))^{log\, i} \in log$); and the initial unbounded universal quantifiers may obviously be merged into one.

Moreover, ξ_m is also equivalent to $\forall i \exists p \in \mathcal{E}_i \Gamma_m(p, i)$. The right-to-left direction is trivial: for any i the $p \in \mathcal{E}_i$ satisfying $\Gamma(p, i)$ will be good for all sequences of witnesses. For the other direction, given a fixed i, there is always an "optimal" sequence of witnesses $\langle x_{j,\psi,\underline{\bar{a}}} \rangle$, i.e. one such that if there is any $x \leq exp^{(3)}j$ for which $\psi'(a_1, \ldots, a_r, x)$ holds, then $x_{j,\psi,\underline{\bar{a}}}$ is such an x. Now, ξ_m gives us a $p \in \mathcal{E}_i$ which works for this "optimal" sequence. One easily checks that p must satisfy $\Gamma_m(p, i)$. $\square$

We also introduce the sentences Ψ_n, for $n \in \omega$:

$$\Psi_n := \bigvee_{\epsilon \in \{0,1\}^{n+1}} \Big(\bigwedge_{r \leq n,\ \epsilon_r = 0} \neg \phi_r \ \wedge \bigwedge_{r \leq n\ \epsilon_r = 1} (\phi_r \wedge \neg \sigma_{r,(\epsilon \restriction r)}) \Big).$$

Finally, we define our theory T by:

$$T := T_0 + \{\Psi_n : n \in \omega\}.$$

Obviously, T is a recursive theory. We will now prove our main theorem, which shows, among others, that T axiomatizes a certain class of models of T_0 in which the set of elements definable by Δ_0 formulae of restricted complexity is cofinal in the set of all Δ_0 definable elements:

Theorem 4.3 *(a) T is consistent.*

(b) For any (not necessarily countable) $\mathbf{M} \models T$, $\mathbf{M}^{\Sigma_1}$ *is recursively reducible to* $\mathbf{M}^{\exists_m}$.

(c) In any (not necessarily countable) $\mathbf{M} \models T$, $E_{m+1} \wedge U_{m+1}(\mathbf{M})$ *is cofinal in* $\Delta_0(\mathbf{M})$.

Proof. We first prove (a). The proof is an inductive construction based on repeated application of theorem 2.11.

In the initial step, take an arbitrary countable model $\mathbf{M}$ of T_0. Consider $\sigma_0 = \sigma_{0,\emptyset}$ and put

$$\mathbf{M}'_0 = \begin{cases} \mathbf{M}[P_0], & if\ \mathbf{M} \models \sigma_0; \\ \mathbf{M}, & otherwise, \end{cases}$$

where P_0 is as in theorem 2.11 for $k = 1$ and $\mathcal{F}_0$ defined as

$$\{\tau_n(x) : n \in \omega\} \cup \{\neg\phi_0\} \cup \{\chi\},$$

where χ should be treated as a relativization of the original χ to some $exp^{(3)}(\cdot)$ true in $\mathbf{M}$ (note that the existence of P_0 follows from the fact that $\mathbf{M} \models \sigma_0$ via theorem 2.11 and lemma 3.1). Also let $\epsilon_0 = 0$ in the former and $\epsilon_0 = 1$ in the latter case.

$\mathbf{M}'_0$ clearly satisfies $I\Delta_0 + \Omega_3$ and $\{\tau_n : n \in \omega\} + \chi$ (either by our assumptions on $\mathbf{M}$ or by the choice of $\mathcal{F}_0$). Furthermore, by lemma 2.10, $\mathbf{M} \models \sigma_0$ implies $\mathbf{M}[P_0] \models \neg\phi_0$. On the other hand, in all models of $\{\tau_n : n \in \omega\}$, $\neg\phi_0$ implies σ_0, because of the validity of a suitable τ_N. Hence either

$\epsilon_0 = 0$ and $\mathbf{M}'_0 \models \neg\phi_0$

or

$\varepsilon_0 = 1$ and $\mathbf{M}'_0 \models \phi_0 \wedge \neg\sigma_0$.

In other words, $\mathbf{M}'_0 \models \Psi_0$. Thus, we always have $\mathbf{M}'_0 \models (T_0 \setminus B\Sigma_1) + \Psi_0$. By passing to a 1–elementary extension of the same height if necessary (see the beginning of the proof of lemma 4.1), we may obtain a model $\mathbf{M}_0$ satisfying $T_0 + \Psi_0$.

Proceeding inductively, assume that we are given a model $\mathbf{M}_n$ satisfying $T_0 + \Psi_n$. Similarly as in the initial step, consider $\sigma_{n+1} = \sigma_{n+1,\epsilon}$ for the sequence $\epsilon = \langle \epsilon_0, \ldots, \epsilon_n \rangle$ determined uniquely in view of $\mathbf{M}_n \models \Psi_n$. Put

$$\mathbf{M}'_{n+1} = \begin{cases} \mathbf{M}_n[P_{n+1}], & if\ \mathbf{M} \models \sigma_{n+1}; \\ \mathbf{M}_n, & otherwise, \end{cases}$$

where P_{n+1} is as in theorem 2.11 for $k = 1$ and $\mathcal{F}_{n+1}$ defined as

$$\{\tau_n(x) : n \in \omega\} \cup \{\neg\phi_r : r \leq n, \epsilon_r = 0\}$$
$$\cup \{\neg\sigma_{r,(\epsilon\restriction r)} : r \leq n, \epsilon_r = 1\} \cup \{\neg\phi_{n+1}\} \cup \{\chi\},$$

where χ and the $\neg\sigma$'s should again be treated as true relativizations to some $exp^{(3)}(\cdot)$ (note as previously that the existence of P_{n+1} follows from $\mathbf{M}_n \models \sigma_{n+1}$ via theorem 2.11 and lemma 3.1). Define $\epsilon_{n+1} = 0$ in the former and $\epsilon_{n+1} = 1$ in the latter case.

Again, it is clear that $\mathbf{M}'_{n+1}$ satisfies $I\Delta_0 + \Omega_3$ and $\{\tau_n : n \in \omega\} + \chi$. As in the initial step, we get either

$\epsilon_{n+1} = 0$ and $\mathbf{M}'_{n+1} \models \neg\phi_{n+1}$

or

$\epsilon_{n+1} = 1$ and $\mathbf{M}'_{n+1} \models \phi_{n+1} \wedge \neg\sigma_{n+1}$.

We now check that $\mathbf{M}'_{n+1} \models \Psi_{n+1}$. This is obvious if $\epsilon_{n+1} = 1$, so assume $\epsilon_{n+1} = 0$ and thus $\mathbf{M}'_{n+1} = \mathbf{M}_n[P_{n+1}]$. For a given $r \leq n$, if $\epsilon_r = 0$, then $\mathbf{M}'_{n+1} \models \neg\phi_r$ as required, since $P_{n+1} \models \neg\phi_k$. On the other hand, if $\epsilon_r = 1$, then $\mathbf{M}'_{n+1} \models \neg\sigma_{r,(\epsilon\restriction r)}$, since P_{n+1} sets a suitable relativization of $\neg\sigma_{r,(\epsilon\restriction r)}$ to "True". But this also means $\mathbf{M}'_{n+1} \models \phi_r$, as $\neg\phi_r$ would imply $\sigma_{r,(\epsilon\restriction r)}$ in view of a suitable τ_N. Thus, in either case, $\mathbf{M}'_{n+1} \models \Psi_{n+1}$.

As before, we may pass to a 1–elementary extension if necessary to get a model $\mathbf{M}_{n+1}$ satisfying $T_0 + \Psi_n$. Since Ψ_n clearly implies Ψ_k for $k < n$, this shows that every finite subtheory of T is consistent. By compactness, T itself is also consistent, which ends the proof of (a).

To prove (b), let $\mathbf{M}$ be an arbitrary model of T. Let the infinite binary sequence ϵ be the unique extension of the sequences given by the Ψ_n's. Then for each $n \in \omega$ we have

$$(*)\, \mathbf{M} \models \phi_n \equiv \neg\sigma_{n,(\varepsilon\restriction n)}.$$

For, just as in the proof of (a), $\neg\phi_n$ implies $\sigma_{n,(\epsilon\restriction n)}$ since $\mathbf{M} \models \{\tau_n : n \in \omega\}$, while ϕ_n yields $\epsilon_n = 1$, whence we have $\neg\sigma_{n,(\epsilon\restriction n)}$ because of Ψ_n.

¿From $(*)$ we obtain a recursive reduction of Σ_1 truth about $\mathbf{M}$ to $\exists_m$ truth about $\mathbf{M}$. Indeed, knowing $(\epsilon \restriction n)$ and knowing whether $\sigma_{n,(\epsilon\restriction n)}$ is true we deduce whether ϕ_n is true, whence we deduce $(\epsilon \restriction n + 1)$ and so on: we recover the Σ_1 truth from the $\forall_m$ truth step by step.

For a proof of (c), suppose that $a \in \Delta_0(\mathbf{M})$. In other words, $\mathbf{M} \models \varphi(a)$, where $\varphi(x) \in \Delta_0$ and $\mathbf{M} \models \exists! x \varphi^x(x)$. Thus, $\exists x \varphi^x(x)$ is a Σ_1 sentence, say ϕ_n, true in $\mathbf{M}$. Let i be such that $exp^{(3)} i < a$, so that we have $\mathbf{M} \models \neg\exists x < exp^{(3)} i\, \varphi^x(x)$. Using an appropriate τ_N (recall that $\mathbf{M} \models \{\tau_n :\ n \in \omega\}$), we can find a $p \in \mathcal{E}_i$ such that in $\mathbf{M}$ we have:

$$\Gamma_m(p, i) \wedge \forall l, l_1 < (log\, i)/2 \forall x (x = \tau_{l,l_1} \Rightarrow True(p, i, x))$$
$$\wedge \bigwedge\nolimits_{r<n} \big(\epsilon_r = 0 \Rightarrow p \models \neg\phi_r\big) \wedge p \models \neg\phi_n,$$

where ϵ is the sequence given by Ψ_n.

Hence

$$\mathbf{M} \models \bar{\sigma}_{n,\epsilon}(i),$$

where $\bar{\sigma}_{n,\epsilon}$ is obtained from $\sigma_{n,\epsilon}$ (the standard version, not necessarily the one discussed in lemma 4.2) by deleting the universal quantifier $\forall i$.

We have proved that for any i, $\neg\exists x < exp^{(3)} i\, \varphi^x(x)$ implies $\bar{\sigma}_{n,\epsilon}(i)$. However, we have $\mathbf{M} \models \neg\sigma_{n,\epsilon}$ since $\mathbf{M} \models \phi_n$. It follows from $\neg\sigma_{n,\epsilon}$ that there exists a number i_0 such that $\mathbf{M} \models \neg\bar{\sigma}_{n,\epsilon}(i_0)$.

Let b_0 be $exp^{(4)}(2 \cdot log\, i_0)$ (thus, b_0 is large enough to be a bound for all the quantifiers in $\bar{\sigma}_{n,\epsilon}(i_0)$), and let $b > b_0$ be the smallest element of $\mathbf{M}$ which is large enough to be a bound for all the quantifiers in $b_0 = exp^{(4)}(2 \cdot log\, i_0)$. Now, b is the smallest element satisfying the E_{m+1} formula

$$\exists i_0 < b\, \exists b_0 < b\big((b_0 = exp^{(4)}(2 \cdot log\, i_0))^b \wedge \neg(\bar{\sigma}_{n,\epsilon}(i_0))^{b_0}\big),$$

so it is definable in $\mathbf{M}$ by an $E_{m+1} \wedge U_{m+1}$ formula. Furthermore, $b > a$. This proves that (c) holds. $\Box$

We conclude this paper with a remark on Σ_1–definability of $\mathbb{N}$ in models of $I\Delta_0 + \Omega_1$ — more precisely, on its relation to the question whether elements definable by Σ_1 formulae of some fixed complexity are cofinal in a given model.

Let $\mathbf{M} \models I\Delta_0 + \Omega_1$ and assume that the set $\exists_r(\mathbf{M})$ is cofinal in $\mathbf{M}$. We claim that if $\mathbf{M}^{\exists_r}$ has a code $a \in \mathbf{M}$, then $\mathbb{N}$ is Σ_1 definable in $\mathbf{M}$ with a as a parameter. For, given an enumeration $\langle \varphi_n : n \in \mathbb{N} \rangle$ of $\exists_r$ sentences, let $\psi(x)$ be the formula

$$\exists y \forall z < x\big(\varphi_z \in a \Rightarrow Sat_r^y(\varphi_z)\big).$$

Clearly, it follows from the cofinality of $\exists_r(\mathbf{M})$ that $\psi(x)$ defines $\mathbb{N}$ in $\mathbf{M}$.

We have thus proved one half of the following proposition (the other follows easily by a standard argument):

Proposition 4.4 *Assume that $\mathbb{N}$ is not Σ_1 definable (with parameters) in $\mathbf{M}$. Then for any r: $\exists_r(\mathbf{M})$ is cofinal in $\mathbf{M}$ iff $\exists_r$ truth is not codable in $\mathbf{M}$.*

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