Well-behaved Principles Alternative to Bounded Induction

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Abstract

We introduce some Π_1–expressible combinatorial principles which may be treated as axioms for some bounded arithmetic theories. The principles, denoted $Sk(\Sigma^b_n, length\ log^k)$ and $Sk(\Sigma^b_n, depth\ log^k)$ (where 'Sk' stands for 'Skolem'), are related to the consistency of Σ^b_n–induction: for instance, they provide models for Σ^b_n–induction. However, the consistency is expressed indirectly, via the existence of evaluations for sequences of terms. The evaluations do not have to satisfy Σ^b_n–induction, but must determine the truth value of Σ^b_n statements.

Our principles have the property that $Sk(\Sigma^b_n, depth\ log^k)$ proves $Sk(\Sigma^b_{n+1}, length\ log^k)$. Additionally, $Sk(\Sigma^b_n, length\ log^{k-2})$ proves $Sk(\Sigma^b_{n+1}, length\ log^k)$. Thus, some provability is involved where conservativity is known in the case of Σ^b_n induction on an initial segment and induction for higher Σ^b_m classes on smaller segments.

1 Introduction

Bounded arithmetic theories are normally axiomatized using induction principles for various classes of bounded formulae, such as Buss' Σ^b_n classes (see

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e.g. [HP]). Some of these these principles are additionally restricted to proper initial segments of models. For example, Buss' theory S^n_2 is $I\Sigma^b_n|log$, induction for Σ^b_n formulae which is restricted to the logarithmic part of a model. In general, it is not known whether induction principles restricted in this way can be derived from full induction for slightly smaller classes of formulae. In particular, the question of whether $I\Sigma^b_{n+1}|log$ can be derived from $I\Sigma^b_n$ is an outstanding open problem.

On the other hand, what is known about these induction principles is that there are interesting conservativity relationships. The most famous result here is that $I\Sigma^b_{n+1}|log$ is $\forall\Sigma^b_{n+1}$ conservative over $I\Sigma^b_n$. This has been generalized by A. Beckmann ([B]) and C. Pollett ([P]) to the case of $I\Sigma^b_{n+1}|log^{k+1}$ in place of $I\Sigma^b_{n+1}|log$ and $I\Sigma^b_n|log^k$ in place of $I\Sigma^b_n$ (where log^k is the k–th iteration of logarithm). Some changes in the assumptions were needed to obtain the generalization: in particular, fuctions of slightly higher growth rate than the standard ω_1 had to be allowed, and standard Σ^b_n classes had to be replaced by their prenex versions.

In the present paper, we propose and discuss a different class of principles. Our theories will contain only some fixed small amount of induction; their most important component will be a certain combinatorial principle, denoted Sk. $Sk(\Sigma^b_n, depth\ log^k)$ will stand for the version of Sk restricted to Σ^b_n formulae and sequences of terms of depth in the log^k part of a model, and $Sk(\Sigma^b_n, length\ log^k)$ for the version of Sk restricted to Σ^b_n formulae and sequences of terms whose length is in the log^k part of a model (see section 3 and the beginning of section 4 for precise definitions).

After introducing some basic definitions and constructions, we try to explain the link between the Sk principles and bounded induction (section 4). We then go on to prove our main result, which states that the Sk principles are, in a sense, better–behaved than induction principles: $Sk(\Sigma^b_n, depth\ log^k)$ suffices to prove $Sk(\Sigma^b_{n+1}, length\ log^k)$, and furthermore, also $Sk(\Sigma^b_n, length\ log^{k-2})$ proves $Sk(\Sigma^b_{n+1}, length\ log^k)$. $Sk(\Sigma^b_n, length\ log^{k-2})$ and $Sk(\Sigma^b_{n+1}, length\ log^k)$ are related to $I\Sigma^b_n|log^{k-2}$ and $I\Sigma^b_{n+1}|log^k$ respectively (theorem 4.3 below). Via this relationship it follows that a counterpart of less induction restricted to a larger cut, expressed by $Sk(\Sigma^b_n, length\ log^{k-2})$, implies a counterpart of more induction restricted to a smaller cut, expressed by $Sk(\Sigma^b_{n+1}, length\ log^k)$. For the theories $I\Sigma^b_n|log^{k-2}$ and $I\Sigma^b_{n+1}|log^k$ themselves this is an open question — see the diagram following corollary 5.2. As in the case of the Beckmann–Pollett results, we need to restrict ourselves to prenex Σ^b_n classes, and to allow some functions which grow slighly faster than ω_1 (more specifically, we have to allow ω_K for some K which depends on k.)

Our main notion is the notion of a Σ^b_n evaluation. Given a sequence of closed terms Λ, an evaluation on Λ is a function which assigns logical values to some atomic sentences with terms from Λ. An evaluation is Σ_n^b if the information it provides makes it possible to decide which Σ_n^b sentences with terms in Λ are to be considered true and which false.

2 Preliminaries

Some notational conventions:

The symbol log stands for the discrete–valued binary logarithm function; $exp(m)$ is 2^m. A superscript over a function symbol (say, log^k) denotes iteration. For a model $\mathbf{M}$, $log^k(\mathbf{M})$ consists of those elements of $\mathbf{M}$ for which exp^k exists. A "bar" always denotes a tuple, and if $\bar{t}$ is $\langle t_1, \ldots, t_l \rangle$, then $\bar{h}_{\bar{t}}$ is $\langle h_{t_1}, \ldots h_{t_l} \rangle$. If Λ is a sequence of terms, $\bar{t} \in \Lambda$ means all of $t_1, \ldots, t_l$ appear in Λ.

We adopt the coding of sets and sequences in bounded arithmetic developed in [HP]. Also the notion of length $lh(\Lambda)$ of a sequence Λ is the one defined in [HP] for bounded arithmetic. If $lh(\Lambda)$ is in $log(\mathbf{M})$ for a model $\mathbf{M}$ of bounded arithmetic, then functions from Λ into $\{0, 1\}$ can be coded in $\mathbf{M}$ as subsets of size $lh(\Lambda)$ of $\Lambda \times \{0, 1\}$ (see [S]). Here we shall use a different coding of such functions. If $\Lambda = \langle t_1, \ldots, t_l \rangle$, then a function f from Λ into $\{0, 1\}$ is given by the pair $\langle \Lambda, p \rangle$, where p is a function from $\{1, \ldots, l\}$ into $\{0, 1\}$ (an object of size 2^l), with $p(i)$ intended to code $f(t_i)$. Whenever Λ is fixed, we may simply identify f with p.

Our base language, L_K (for some natural number K), contains 0, 1, $+$, $<$, $\times$, $|\cdot|$ (the length function symbol), and, for $i \leq K$, the symbols $\#_i$ for the smash functions [2]. We assume that some appropriate Gödel numbering of L_K formulae has been fixed; we shall identify the formulae with their Gödel numbers.

To this language we add function symbols s^φ for all L_K formulae φ in prenex normal form which begin with an existential quantifier. The symbol s^φ is intended to stand for a Skolem function for the first existential quantifier in φ. That is, given an L_K formula $\varphi(\bar{x})$ in normal form, if $\varphi(\bar{x})$ is $\exists y \psi(\bar{x}, y)$, then s^φ is a function symbol of arity $1 + lh(\bar{x})$, and $s^\varphi(\bar{t})$ is intended to be some y which satisfies $\varphi'(\bar{t}, y)$, if such a y exists.

[2] The length $|x|$ of x is $\lceil log(x+1) \rceil$. The smash functions are defined by: $x \#_2 y = exp(|x| \cdot |y|); x \#_{m+1} y = exp(|x| \#_m |y|)$. A related family of functions is defined by: $\omega_1(x) = x^{|x|}; \omega_{m+1}(x) = exp(\omega_m(|x|))$. Note that $\omega_m(x)$ is roughly $x \#_{m+1} x$.

We include the symbols of L_K among the s^φ's: for example, $t_1 + t_2$ may be treated as $s^{\exists z(z=x+y)}(t_1, t_2)$.

Whenever we speak of a formula $\varphi(\bar{t})$, it is assumed that $\varphi(\bar{x})$ itself is an L_K formula, although the terms $\bar{t}$ do not have to be terms of L_K.

We have to encode our extended language in arithmetic. We use even numbers to enumerate terms of the form $s^\varphi(\bar{t})$, and odd numbers for a special enumeration of numerals. More precisely, we let the number $2\langle \varphi(\bar{x}), \bar{t} \rangle$ correspond to $s^\varphi(\bar{t})$ (it is assumed that some Gödel numbering of the formulae of L_K has already been fixed), and we let $2k+1$ correspond to a numeral for k ($2k+1$ will be referred to as $\underline{k}$). From now on, we identify terms with their numbers.

We fix a standard natural number N, which will play the role of a parameter. Many of our definitions depend on N, and often we will consider only formulae $< N$ (more precisely, formulae of the form $\varphi(\bar{t})$, where $\bar{t}$ is a tuple of terms and (the Gödel number of) $\varphi(\bar{x})$ is smaller than N). [3] We also fix the numbers $k \geq 1$ (this will determine which iteration of the logarithm function we work with), K (in order to fix L_K), and n (in order to fix Σ^b_n). Our definition of Σ^b_n differs slightly from the one most commonly used. For one thing, we allow quantifiers bounded by any terms of the language L_K, and thus also by $\#_i$, even if i is not equal to 2. For another, we work with prenex Σ^b_n classes, instead of Σ^b_n in the more usual, broader sense (see [HP]). In the standard model, every Σ^b_n formula in the broader sense is equivalent to a prenex Σ^b_n formula, but some theories we consider might not be able to prove this equivalence.

A term is Σ^b_n if it is $s^\varphi(\bar{t})$ for $\varphi \in \Sigma^b_n$. Numerals are considered Σ^b_n terms for any n. The depth of a term is defined in the natural inductive way, with the exception that all numerals are considered to have depth 0.

Some of our constructions and definitions require a limited amount of induction. Therefore, we assume that all models we deal with satisfy $I\Sigma^b_{n_0}$ for some appropriate fixed small number n_0. We write T_0 to denote the theory $I\Sigma^b_{n_0}$ in the language L_K. Thus, whenever we speak of a model $\mathbf{M}$, it is assumed that $\mathbf{M} \models T_0$ — and that $\mathbf{M}$ is nonstandard. The universe of $\mathbf{M}$ will be denoted by M.

The results of sections 4 and 5 which involve the parameter n hold for n "sufficiently large with respect to n_0".

We shall consider various sequences of closed terms. About such a sequence

[3] We make this restriction for the sake of technical simplicity. We could work without it, and use appropriate universal formulae for Σ^b_n where necessary. The restriction is related to the fact that we may axiomatize $I\Sigma^b_n$ by instances of induction referring to formulae $< N$ — we make use of this fact in theorem 4.3.

Λ we shall always assume that if a term of the form $s^{\varphi}(\bar{t})$ appears in Λ, then all terms in $\bar{t}$ also do, and moreover, that they have smaller indices in Λ than $s^{\varphi}(\bar{t})$. Given a Λ, we denote by $top(\Lambda)$ the largest number h such that the numeral $\underline{h}$ is in Λ.

¿From now on, whenever we deal with a sequence of terms Λ and a model $\mathbf{M}$ of bounded arithmetic, we shall assume that $lh(\Lambda)$ is in $log(\mathbf{M})$.

Given a tuple of variables $\langle x_1, \ldots, x_m \rangle$, the collection of *simple atomic formulae* over $\langle x_1, \ldots, x_m \rangle$ consists of $x_i = x_j, x_i < x_j, x_i = 0, x_i + x_j = x_l, x_i = |x_j|$ etc. for other symbols of L_K ($1 \leq i, j, l \leq m$; basically, simple atomic formulae are those which would still be considered atomic if the vocabulary was relational). Any open formula over $\langle x_1, \ldots, x_m \rangle$ which does not contain nested terms (such as $(x_i + x_j) \times x_l$) is a boolean combination of simple atomic formulae. For a sequence of closed terms Λ, let the collection $\mathcal{E}(\Lambda)$ of simple atomic sentences over Λ consist of all sentences obtained by substituting terms from Λ for the x_i's in simple atomic formulae. Note that $lh(\mathcal{E}(\Lambda))$ is polynomial in $lh(\Lambda)$.

3 Evaluations

Suppose a sequence of closed terms Λ is given. For $\varphi(\bar{x})$ in normal form, $\bar{t} \in \Lambda$, we define the notion that Λ is *good enough* (g.e.) for $\langle \varphi, \bar{t} \rangle$ by induction on φ. Λ is always g.e. for $\langle \varphi, \bar{t} \rangle$ if φ is simple atomic. If $\varphi(\bar{x})$ is $f_1(\bar{x}) = f_2(\bar{x})$ where f_1 and/or f_2 are nested terms, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if Λ contains $s^{\exists y(y=f_i(\bar{x}))}(\bar{t})$ for the appropriate i's (similarly for '$<$' in place of '$=$'). If φ is $\exists y \varphi'(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^{\varphi}(\bar{t}) \in \Lambda$ and Λ is g.e. for $\langle \varphi', \bar{t}^\frown s^{\varphi}(\bar{t}) \rangle$. Finally, if φ is $\forall y \tilde{\varphi}(\bar{x}, y)$, then Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if $s^{\exists y \neg \tilde{\varphi}(\bar{x}, y)}(\bar{t}) \in \Lambda$ (where $\exists y \neg \tilde{\varphi}$ is the normal form of $\neg \varphi$) and Λ is g.e. for $\langle \tilde{\varphi}, \bar{t}^\frown s^{\exists y \neg \tilde{\varphi}(\bar{x}, y)}(\bar{t}) \rangle$.

The idea is that Λ is g.e. for $\langle \varphi, \bar{t} \rangle$ if it contains enough appropriate Skolem terms so that assigning a logical value to $\varphi(\bar{t})$ based on an evaluation on Λ (defined below) makes sense.

Definition 3.1 [4]

Let $p : \mathcal{E}(\Lambda) \longrightarrow \{0, 1\}$ map every axiom of equality in $\mathcal{E}(\Lambda)$ to 1. We think of p as assigning a logical value to sentences in $\mathcal{E}(\Lambda)$.

Let $\bar{t} \in \Lambda$. We define the relation $p \models \varphi(\bar{t})$ for $\varphi(\bar{x})$ in normal form by induction:

[4] See also [A1], [A2], [A3], [AZ1], [AZ2], [S].

(i) $p \models \varphi(\bar{t})$ *iff* $p(\varphi(\bar{t})) = 1$ *for* $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$*, and the relation* $p \models \varphi$ *behaves in the natural way with respect to boolean combinations of formulae in* $\mathcal{E}(\Lambda)$*;*

(ii) *if* $\varphi(\bar{t})$ *is atomic but contains nested terms, then* $p \models \varphi(\bar{t})$ *iff:* Λ *is g.e. for* $\langle\varphi, \bar{t}\rangle$*, and if* $\psi(\bar{t}, \bar{s}(\bar{t}))$ *is the formula obtained by substituting the Skolem terms for the nested terms in* $\varphi(\bar{t})$*, then* $p \models \psi(\bar{t}, \bar{s}(\bar{t}))$*,*

(iii) *if* φ *is* $\exists y \varphi'(\bar{x}, y)$*, then* $p \models \varphi(\bar{t})$ *iff* Λ *is g.e. for* $\langle\varphi, \bar{t}\rangle$ *and* $p \models \varphi'(\bar{t}, s^{\varphi}(\bar{t}))$*,*

(iv) *if* φ *is* $\forall y \tilde{\varphi}(\bar{x}, y)$*, then* $p \models \varphi(\bar{t})$ *iff for all* $t \in \Lambda$ *such that* Λ *is g.e. for* $\langle\tilde{\varphi}, \bar{t}^\frown t\rangle$*,* $p \models \tilde{\varphi}(\bar{t}, t)$*.*

Definition 3.2 *Let* Λ *be given. A function* $p : \mathcal{E}(\Lambda) \longrightarrow \{0, 1\}$ *is called a* Σ^b_n *evaluation on* Λ *if the following holds:*

(1) *For every* Σ^b_n *formula* $\varphi(\bar{x})$*,* $\varphi < N$*, and every* $\bar{t} \in \Lambda$ *of the appropriate length, if* Λ *is g.e. for* $\langle\varphi, \bar{t}\rangle$*, then*

$$p \models \varphi(\bar{t}) \text{ or } p \models \neg\varphi(\bar{t});$$

(2) *if* $\varphi(\bar{t})$*,* $\varphi < N$*,* $\bar{t} \in \Lambda$*, is an instance of an axiom of* T_0 *or if* $\bar{t}$ *are numerals and* $\varphi(\bar{t})$ *is a true* $\Sigma^b_{n_0}$ *or* $\Pi^b_{n_0}$ *sentence, then assuming* Λ *is g.e. for* $\langle\varphi, \bar{t}\rangle$*,* $p \models \varphi(\bar{t})$*.*

An "evaluation on Λ" is simply a Σ^b_n evaluation on Λ for some $n \geq n_0$.

Definition 3.3 *Let* p, p' *be evaluations on* Λ*,* Λ' *respectively. We say that* p' *extends* p *if* $\Lambda \subseteq \Lambda'$ *and* $p \subseteq p'$*.*

Proposition 3.4 *If* p, p' *are* Σ^b_n *evaluations on on* Λ*,* Λ' *respectively and* p' *extends* p*, then and for any* $\varphi \in \Sigma^b_n$*,* $\varphi < N$*,* $\bar{t} \in \Lambda$*, if* Λ *is good enough for* $\langle\varphi, \bar{t}\rangle$ *then*

$$p \models \varphi(\bar{t}) \text{ iff } p' \models \varphi(\bar{t}).$$

Proof. A simple inductive argument. □

Let $\mathbf{M}$ be a model and let a sequence of closed terms Λ be given. Denote by $TERM(\Lambda)$ the set of all terms of those terms of standard depth whose subterms of depth 0 (i.e. numerals) are in Λ. Assume that $TERM(\Lambda) \subseteq \Lambda$. In that case, every Σ^b_n evaluation p on Λ determines a structure $\mathbf{M}(p)$ which "agrees" with p about which Σ^b_n formulae smaller than N are satisfied.

$\mathbf{M}(p)$ is constructed as follows. Let the relation $\sim$ on $TERM(\Lambda)$ be defined by $t \sim t' \iff p \models (t = t')$. Since p is an evaluation, $\sim$ is an equivalence relation and a congruence with respect to the arithmetical operations. Thus, we can define the universe of $\mathbf{M}(p)$ as $TERM(\Lambda)/\sim$; the operations of $\mathbf{M}(p)$ are defined in the obvious way. It now follows from the definition of evaluation that for any $\Sigma_n^b \cup \Pi_n^b$ formula $\varphi < N$ and any tuple $\bar{t} = \langle t_0, \ldots, t_m \rangle \in \Lambda$,

$$\mathbf{M}(p) \models \varphi([t_0], \ldots, [t_m]) \text{ iff } p \models \varphi(\bar{t}),$$

where $[t_i]$ denotes the $\sim$–equivalence class of t_i.

A convenient way to obtain evaluations on a sequence Λ is to use *Skolem hulls*. A hull on Λ is a sequence $H = \langle h_t : t \in \Lambda \rangle$ of elements of M, where the element h_t is thought of as an interpretation of the term t. It is assumed that for every numeral $\underline{k} \in \Lambda$, $h_{\underline{k}} = k$. The satisfaction relation $H \models \varphi(\bar{t})$ is defined similarly to $p \models \varphi(\bar{t})$. We take

$$H \models \varphi(\bar{t}) \text{ iff } \mathbf{M} \models \varphi(\bar{h}_t)$$

for $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$, and later proceed inductively just as in definition 3.1. A *Skolem* Σ_n^b *hull* is then defined analogously to a Σ_n^b evaluation.

Observe that if H is a Skolem Σ_n^b hull on Λ, then the function p_H defined for $\varphi(\bar{t}) \in \mathcal{E}(\Lambda)$ by the clause

$$p_H(\varphi(\bar{t})) = 1 \text{ iff } H \models \varphi(\bar{t})$$

is a Σ_n^b evaluation on Λ. We say that p_H is *isomorphic* with H.

A *true Skolem* Σ_n^b *hull* on Λ is a Skolem Σ_n^b hull H on Λ which additionally satisfies the following: for every formula $\varphi(\bar{x}) < N$ which is at most Π_n^b and starts with a universal quantifier, and every $\bar{t} \in \Lambda$, if $H \models \varphi(\bar{t})$, then $\varphi(\bar{h}_{\bar{t}})$ is true (in $\mathbf{M}$).

4 The *Sk* principles

In this section, we introduce the *Sk* principles and the theories they axiomatize.

Definition 4.1 *Let Λ be a sequence of closed terms. We say that Λ is of depth i if all terms in Λ have depth $\leq i$.*

Let $Sk(\Sigma_n^b, \textit{depth } log^k)$ be the theory axiomatized by T_0 and the following sentence:

"For every $i \in log^k$ and every Λ of depth at most i there exists a Σ_n^b evaluation on Λ."

Also let $Sk(\Sigma_n^b, length\ log^k)$ be the theory axiomatized by T_0 and the following sentence:

"For every $i \in log^k$ and every Λ of length at most i and depth at most $log(i)$ there exists a Σ_n^b evaluation on Λ."

Observe that both $Sk(\Sigma_n^b, length\ log^k)$ and $Sk(\Sigma_n^b, depth\ log^k)$ are Π_1–axiomatizable theories. It is clear that T_0 is Π_1–axiomatizable, but perhaps less obvious that the additional principles can also be formulated as Π_1 statements. Let us argue the case of the depth principle (the other is quite similar). We may express this principle by a formula which begins with universal quantifiers for $y = exp^k(i)$, for Λ, and for $z = 2^{\pi(lh(\Lambda))}$ where π is some standard polynomial to be specified below. We claim that the rest of the formula may then be bounded. Being of depth i is certainly definable by a bounded formula, so the main question is whether the existential quantifier for evaluations can be bounded. Any evaluation p on Λ is a pair $\langle \mathcal{E}(\Lambda), p' \rangle$, where p' is a function from $\{1, \ldots, lh(\mathcal{E}(\Lambda))\}$ into $\{0, 1\}$. Since $lh(\mathcal{E}(\Lambda))$ is polynomial in $lh(\Lambda)$, we may take π to be a polynomial such that $\mathcal{E}(\Lambda)$ and p' are both bounded by $2^{\pi(lh(\Lambda))}$. As $\langle a, b \rangle \leq 2(a+b)^2$, the claim now follows.

Actually, we may assume that both $Sk(\Sigma_n^b, length\ log^k)$ and $Sk(\Sigma_n^b, depth\ log^k)$ are even $\forall\Sigma_{n_0+1}^b$–axiomatizable. To see this, we only need to check that our principles are $\forall\Sigma_{n_0+1}^b$. An examination of definition 3.1 reveals that the relation "$p \models \varphi$" is definable by a fixed bounded formula, so we may assume that it is $\Sigma_{n_0}^b$–definable. It follows that the property of being a Σ_n^b evaluation is also definable by a bounded formula of fixed (i.e. independent of n) complexity. Here we are not allowed to assume that this is a $\Sigma_{n_0}^b$ property, as part (2) of the definition of a Σ_n^b evaluation (def. 3.2) contains some implications with $\Sigma_{n_0}^b$ antecedents. However, there are no obstacles to assuming that being a Σ_n^b evaluation is $\Sigma_{n_0+1}^b$. Hence, the statement that a Σ_n^b evaluation exists on every appropriate Λ is, as required, $\forall\Sigma_{n_0+1}^b$.

The next two theorems show that there is some connection between $Sk(\Sigma_n^b, length\ log^k)$ and induction.

Theorem 4.2 *Assume T_0 and $I\Sigma_n^b|log^k$. Let Λ of length $i \in \log^k$ consist of Σ_n^b terms. Then there exists a true Σ_{n-1}^b hull on Λ.*

Proof. Let $\Lambda = \langle t_0, \ldots, t_l \rangle$. We want to apply $I\Sigma_n^b|log^k$ to the formula "there exists a true Σ_{n-1}^b hull on $\langle t_0, \ldots, t_m \rangle$" for $m \leq l$. The inductive step is quite straightforward, the only difficulty is to check that our formula is indeed Σ_n^b.

The initial existential quantifier can be bounded, since, by our restriction to formulae $< N$, elements of the required hulls can be bounded by $f(top(\Lambda))$

for some fixed L_K–term f. So, it suffices to verify that being a true Skolem Σ_n^b hull is, for sufficiently large n, a Π_n^b property.

Being a Skolem Σ_n^b hull is, just as being a Σ_n^b evaluation (see above), $\Sigma_{n_0+1}^b$–definable. To say that H is a true Skolem Σ_n^b hull, we need to state that H is a Skolem Σ_n^b hull and additionally that it satisfies

$$\forall \bar{t} \in \Lambda \, ((H \models \varphi(\bar{t})) \Rightarrow \varphi(\bar{h}_{\bar{t}})),$$

for a fixed finite number of Π_n^b formulae. □

Thus, $T_0 + I\Sigma_n^b|log^k$ implies $Sk(\Sigma_{n-1}^b, length\ log^k)$ (and hence, if $n \geq n_0 + k$, $I\Sigma_n^b|log^k$ itself implies $Sk(\Sigma_{n-1}^b, length\ log^k)$).

Since $Sk(\Sigma_{n-1}^b, length\ log^k)$ is $\forall\Sigma_{n_0+1}^b$, we may additionally infer

$$I\Sigma_{n-k}^b \vdash Sk(\Sigma_{n-1}^b, length\ log^k)$$

provided $n \geq n_0 + \max(k, 2)$ in the case when $K > k$ (cf [P]).

The relation in the other direction is somewhat more difficult to express. In general terms, we may say that $Sk(\Sigma_n^b, length\ log^k)$ allows us to build a model for $I\Sigma_n^b|log^k$ with an appropriately large k–th logarithm.

In the following theorem, we assume that N is so large that induction axioms for Σ_n^b formulae smaller than N axiomatize $I\Sigma_n^b$. Note that this is always possible, as $I\Sigma_n^b$ is finitely axiomatizable for $n \geq 1$.

Theorem 4.3 *Let* $\mathbf{M} \models Sk(\Sigma_n^b, length\ log^k)$.

Let $l_0, l_1 \in log^k(\mathbf{M})$ *satisfy* $\omega_{K-1}^{\mathbb{N}}(exp^k(l_0)) < exp^k(l_1)$.

Let $\Lambda \in M$ *be such that:* Λ *is of length* i *for some* $i \in log^k$, $TERM(\Lambda) \subseteq \Lambda$, *and* $TERM(\Lambda)$ *contains numerals for:* $0, \ldots, l_1$, $exp^k(j)$ *for any* $j \leq l_1$, *and all standard iterations of the smash functions* $\#_i$ $(i \leq K)$ *applied to* $exp^k(j)$ *for* $j \leq l_1$.

Let p *be a* Σ_n^b *evaluation on* Λ *given by* $Sk(\Sigma_n^b, length\ log^k)$. *Then there exists an initial segment* $\mathfrak{M}$ *of* $\mathbf{M}(p)$ *satisfying* $I\Sigma_n^b|log^k$ *and such that* $l_0 \in log^k(\mathfrak{M})$.

Proof. We first show that $\mathbf{M}(p)$ satisfies $I\Sigma_n^b|l_1$. Consider a Σ_n^b formula $\varphi < N$ (we may restrict ourselves to $\varphi < N$ without loss of generality). Assume that $\mathbf{M}(p) \models \varphi(0)$ and $\mathbf{M}(p) \models \varphi(l) \Rightarrow \varphi(l+1)$ for all $l < l_1$. We thus have $\mathbf{M} \models (p \models \varphi(0))$ and

$$\mathbf{M} \models (p \models \varphi(l) \Rightarrow p \models \varphi(l+1))$$

for $l < l_1$. By $\Sigma^b_{n_0}$ induction in $\mathbf{M}$, it follows that $\mathbf{M} \models (p \models \varphi(l_1))$, whence $\mathbf{M}(p) \models \varphi(l_1)$.

We may now take $\mathfrak{M}$ to be the initial segment $\omega^{-\mathbb{N}}_{K-1}(exp^k(l_1))$ of $\mathbf{M}$ (i.e. $\mathfrak{M}$ consists of those elements $l \in \mathbf{M}(p)$ which satisfy $\omega^n_{K-1}(l) < exp^k(l_1)$ for all $n \in \mathbb{N}$). Clearly, the operations of L_K are well–defined in $\mathfrak{M}$, and $l_0 \in log^k(\mathfrak{M})$ by the assumption that $\omega^{\mathbb{N}}_{K-1}(exp^k(l_0)) < exp^k(l_1)$. Moreover, since $log^k(\mathfrak{M})$ is contained in the segment $[0, l_1)$, we also have $\mathfrak{M} \models I\Sigma^b_n | log^k$. □

5 The main theorem

Our next aim is the proof of our main theorem. All the results of this section require k to be at least 3, since sequences of terms whose length is in log^{k-2} are involved. Recall that our base language L_K contains the symbols $\#_i$ for $i \leq K$.

Theorem 5.1 *Assume* $K > k + 1$. *Then* $Sk(\Sigma^b_n, depth\ log^k) \vdash Sk(\Sigma^b_{n+1}, length\ log^k)$.

In the proof, we make the notational convention that whenever $\exists y \psi(\bar{x}, y)$ is a formula in normal form, then this ψ is denoted by φ'.

Proof. [5]

Assume $Sk(\Sigma^b_n, depth\ log^k)$.

Let $i_0 \in log^{k+1}, l_0 = exp(i_0)$ and let $\hat{\Lambda}$ be of length at most l_0 and contain terms of depth $\leq i_0$. We may assume that $\hat{\Lambda}$ has length exactly l_0 (so that $\hat{\Lambda} = \langle t_0, \ldots, t_{l_0-1} \rangle$) and consists of Σ^b_{n+1} terms. Present $\hat{\Lambda}$ as $\hat{\Lambda}_0 \cup \ldots \cup \hat{\Lambda}_{i_0}$, where $\hat{\Lambda}_m$ consists of those terms in $\hat{\Lambda}$ which have depth m.

Let $j = (l_0)^{i_0}$. Observe that $i_0 \leq |l_0|$, so $j \leq \omega_1(l_0)$. Since $K > k+1$, ω_{k+1} is a total function, so log^k is closed under ω_1. Hence, $j \in log^k$.

Let Λ contain $\hat{\Lambda}_0$, consist of Σ^b_n terms of depth $\leq j$, and be such that for any Σ^b_n formula $\varphi \leq N$ and any $\bar{t} \in \Lambda$ of depth $i < j$ and appropriate length, it holds that $s^{\varphi}(\bar{t}) \in \Lambda$. We additionally assume that $0 \in \Lambda$. Let p be a Σ^b_n evaluation on Λ given by $Sk(\Sigma^b_n, depth\ log^k)$.

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[5] The proof has some ideas in common with [BR], in particular the use of the pigeon hole principle.

Let $u_1, \dots, u_l$ be an enumeration of all pairs $\langle \varphi, \bar{t} \rangle$, $u_{l'} = \langle \varphi_{l'}, \bar{t}_{l'} \rangle$, where $\bar{t}$ is a tuple of terms from $\hat{\Lambda}_0$ of length at most $(N-1)$ and $\varphi < N$ is a Σ^b_{n+1} formula such that $s^{\varphi}(\bar{t}) \in \hat{\Lambda}_1$. Note that there are at most $l_0 - 1$ such pairs.

We define by induction a function $f_1 : \{u_1, \dots, u_l\} \longrightarrow [0, j)$ (along with a sequence $\langle s_1(u_{l'}) : l' \leq l \rangle$ of terms) as follows:

If $\varphi_{l'}$ is a Σ^b_n formula, then $f_1(u_{l'}) = 1$ and $s_1(u_{l'}) = s^{\varphi_{l'}}(\bar{t}_{l'})$.

Otherwise, $f_1(u_{l'})$ is: either the least $1 \leq i < j$ for which there is a Σ^b_n term $s \in \Lambda$ of depth $\leq i$ such that $p \models \varphi'_{l'}(\bar{t}_{l'}, s)$ (in that case, $s_1(u_{l'})$ is some such s); or, if no such i exists, $f_1(u_{l'}) = 0$ and $s_1(u_{l'}) = 0$.

In more detail:

If φ_1 is a Σ^b_n formula, then $f_1(u_1) = 1$ and $s_1(u_1) = s^{\varphi_1}(\bar{t}_1)$ (note that in this case $f_1(u_1)$ is the depth of $s_1(u_1)$). Otherwise, $f_1(u_1)$ is: either the least $1 \leq i < j$ for which there is a Σ^b_n term $s \in \Lambda$ of depth $\leq i$ such that $p \models \varphi'_1(\bar{t}_1, s)$ (in that case, $s_1(u_1)$ is some such s); or, if no such i exists, $f_1(u_1) = 0$ and $s_1(u_1) = 0$ (it then holds that $p \models \neg\varphi_1(\bar{t}_1)$).

Similarly, if φ_2 is a Σ^b_n formula, then $f_1(u_2) = 1$ and $s_1(u_2) = s^{\varphi_2}(\bar{t}_2)$. Otherwise, $f_1(u_2)$ is: either the least $1 \leq i < j$ for which there is a Σ^b_n term $s \in \Lambda$ of depth $\leq i$ such that $p \models \varphi'_2(\bar{t}_2, s)$ (in that case, $s_1(u_1)$ is some such s); or, if no such i exists, $f_1(u_2) = 0$ and $s_1(u_2) = 0$ (it then holds that $p \models \neg\varphi_2(\bar{t}_2)$). Etc.

Note that all the notions required in the definition of f_1, in particular the relation "$p \models \varphi'(\bar{t}, s)$", are definable by bounded (possibly with an extra parameter) formulae of fixed complexity. By choosing a large enough n_0 we may assume that this complexity is suitably less than $\Sigma^b_{n_0}$. Also, f_1 can be coded as described in the preliminaries, i.e. a tuple $\bar{t}$ such that $\langle \varphi, \bar{t} \rangle$ is in the domain of f_1 can be identified with the tuple of indices of the terms $\bar{t}$ in the enumeration of $\hat{\Lambda}_0$. Thus, T_0 will suffice to prove the existence of a code for a function f_1 with the required properties.

We have

$$[0, j) = [0, (l_0)^{i_0-1}) \cup [(l_0)^{i_0-1}, 2(l_0)^{i_0-1}) \cup \ldots \cup [(l_0-1)(l_0)^{i_0-1}, l_0(l_0)^{i_0-1}).$$

As l_0 is quite small (it is certainly in *log*), we may apply the pigeon hole principle to find $r < l_0$ such that the interval $[r(l_0)^{i_0-1}, (r+1)(l_0)^{i_0-1})$ does not contain any value of the function f_1. This is because if all l_0 of the above intervals contained a value of f_1, we could use the code of f_1 to obtain a coded function f from $l_0 - 1$ onto l_0. But the pigeon hole principle of the form

$$\forall f, x\, (f \text{ is not a function from } x-1 \textit{ onto } x)$$

is provable in (a finite fragment of) $I\Delta_0$ — hence we may assume that it is provable in T_0.

So, let $r < l_0$ be such that the interval $[r(l_0)^{i_0-1}, (r+1)(l_0)^{i_0-1})$ does not contain any value of the function f_1.

Let $r_1 = r(l_0)^{i_0-1}, r_1' = (r+1)(l_0)^{i_0-1}$. Let Λ_1 be $\hat{\Lambda}_0 \cup \{s_1(u_{l'}) : l' \leq l, f_1(u_{l'}) < r_1\}$. Also define $\tilde{g}_1 : \hat{\Lambda}_1 \longrightarrow \Lambda_1$ by:

$$\tilde{g}_1(s^{\varphi}(\bar{t})) = \begin{cases} s_1(\bar{t}, \varphi) & \text{if } f_1(\bar{t}, \varphi) < r_1 \\ 0 & \text{otherwise} \end{cases}$$

and $g_1 : \hat{\Lambda}_0 \cup \hat{\Lambda}_1 \longrightarrow \Lambda_1$ as $\tilde{g}_1 \cup id|\hat{\Lambda}_0$.

Note that for $\varphi \in \Sigma_n^b$, $g_1(s^{\varphi}(\bar{t})) = s^{\varphi}(\bar{t})$. For in this case, $f_1(\bar{t}, \varphi) = 1$, whence $f_1(\bar{t}, \varphi) \in [0, (l_0)^{i_0-1})$ and consequently, $f_1(\bar{t}, \varphi) < r_1$.

Now let $u_1, \ldots, u_l$ be an enumeration of all pairs $\langle \varphi, g_1(\bar{t}) \rangle$, $u_{l'} = \langle \varphi_{l'}, g_1(\bar{t}_{l'}) \rangle$, where $\bar{t}$ is a tuple of terms from $\hat{\Lambda}_0 \cup \hat{\Lambda}_1$ of length at most $(N-1)$ and $\varphi < N$ is a Σ_{n+1}^b formula such that $s^{\varphi}(\bar{t}) \in \hat{\Lambda}_2$. Again, there are at most $l_0 - 1$ such pairs.

Let $f_2 : \{u_1, \ldots, u_l\} \longrightarrow [0, j)$ and $\langle s_2(u_{l'}) : l' \leq l \rangle$ be defined by:

If $\varphi_{l'}$ is a Σ_n^b formula, then $f_2(u_{l'})$ is the depth of $s^{\varphi_{l'}}(g_1(\bar{t}_{l'}))$ and $s_2(u_{l'}) = s^{\varphi_{l'}}(g_1(\bar{t}_{l'}))$. Note that the depth of $g_1(\bar{t}_{l'})$ is $< r_1 < j$, whence $s^{\varphi_{l'}}(\bar{t}_{l'}) \in \Lambda$, by our assumption on Λ. Otherwise, $f_2(u_{l'})$ is: either the least $2 \leq i < j$ for which there is a Σ_n^b term $s \in \Lambda$ of depth $\leq i$ such that $p \models \varphi'_{l'}(g_1(\bar{t}_{l'}), s)$ (in that case, $s_2(u_{l'})$ is some such s); or, if no such i exists, $f_2(u_{l'}) = 0$ and $s_2(u_{l'}) = 0$.

We now have

$$\begin{aligned} [r_1, r_1') = [r_1, r_1 + (l_0)^{i_0-2}) \cup [r_1 + (l_0)^{i_0-2}, r_1 + 2(l_0)^{i_0-2}) \\ \cup \ldots \cup [r_1 + (l_0 - 1)(l_0)^{i_0-2}, r_1 + l_0(l_0)^{i_0-2}). \end{aligned}$$

Let $r < l_0$ be such that the interval $[r_1 + r(l_0)^{i_0-2}, r_1 + (r+1)(l_0)^{i_0-2})$ does not contain any value of the function f_2.

Let $r_2 = r_1 + r(l_0)^{i_0-2}, r'_2 = r_1 + (r+1)(l_0)^{i_0-2}$. Let Λ_2 be $\Lambda_1 \cup \{s_2(u_{l'}) : l' \leq l, f_2(u_{l'}) < r_2\}$. Define $\tilde{g}_2 : \hat{\Lambda}_2 \longrightarrow \Lambda_2$ by:

$$\tilde{g}_2(s^{\varphi}(\bar{t})) = \begin{cases} s_2(g_1(\bar{t}), \varphi) \text{ if } f_2(g_1(\bar{t}), \varphi) < r_2 \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

and $g_2 : \hat{\Lambda}_0 \cup \hat{\Lambda}_1 \cup \hat{\Lambda}_2 \longrightarrow \Lambda_2$ as $\tilde{g}_2 \cup g_1$.

Again for $\varphi \in \Sigma^b_n$, $g_2(s^{\varphi}(\bar{t})) = s^{\varphi}(g_1(\bar{t}))$. For in this case, $f_2(g_1(\bar{t}), \varphi)$ is the depth of $g_1(\bar{t})$ plus 1, whence $f_2(g_1(\bar{t}), \varphi) \leq r_1 < r_2$

For $2 < m \leq i_0$, we construct $f_m, r_m, r'_m, \Lambda_m, g_m$, in a similar way. Finally, we take $g : \hat{\Lambda} \longrightarrow \Lambda$ to be $\bigcup_{m \leq i_0} g_m$ and let $\hat{p}$ be the evaluation on $\hat{\Lambda}$ defined by:

$$(*) \;\; \hat{p}(\varphi(\bar{t})) = p(\varphi(g(\bar{t})))$$

(for $\bar{t} \in \hat{\Lambda}$ and φ simple atomic). It remains to show that $\hat{p}$ is a Σ^b_{n+1} evaluation on $\hat{\Lambda}$.

Note that for any Σ^b_n formula $\varphi < N$, if $s^{\varphi}(\bar{t}) \in \hat{\Lambda}$, then $s^{\varphi}(g(\bar{t})) \in \Lambda$, and moreover, $g(s^{\varphi}(\bar{t})) = s^{\varphi}(g(\bar{t}))$. This makes it possible to prove by induction on formula complexity that $(*)$ holds also if $\varphi < N$ is a Σ^b_n formula and $\hat{\Lambda}$ is g.e. for $\langle \varphi, \bar{t} \rangle$ (use the fact that p is a Σ^b_n evaluation in the step for the universal quantifier).

It follows that $\hat{p}$ satisfies part (2) of the definition of a Σ^b_{n+1} evaluation (since p is a Σ^b_n evaluation). As for (1), the most interesting case is when $\varphi \in \Sigma^b_{n+1} \setminus (\Sigma^b_n \cup \Pi^b_n)$. So let $\varphi < N$ be such a formula and assume that $\hat{\Lambda}$ is g.e. for $\langle \varphi, \bar{t} \rangle$. We thus know that in, say, the m–th step of the construction $\langle \varphi, g_{m-1}(\bar{t}) \rangle$ appeared as some $u_{l'}$.

If there was at that point no term $s \in \Lambda$ for which $p \models \varphi'(g_{m-1}(\bar{t}), s)$, then no such term could have appeared later on in the construction, so for any $s \in \Lambda$, $p \models \neg\varphi'(g(\bar{t}), s)$. Then by $(*)$ and definition 3.1, $\hat{p} \models \neg\varphi$.

Otherwise, either $f_m(u_{l'}) < r_m$ or the contrary. In the former case, clearly $p \models \varphi'(g(\bar{t}), g(s^{\varphi}(\bar{t})))$ and hence $\hat{p} \models \varphi(\bar{t})$. In the latter case, since no term $s \in \Lambda$ of depth $< f_m(u_{l'})$ satisfies $p \models \varphi'(g(\bar{t}), s)$, and no term of depth $\geq f_m(u_{l'})$ is in the range of g, we have $\hat{p} \models \neg\varphi(\bar{t})$. This completes the proof of the theorem. $\Box$

Note that the assumption $K > k+1$ was only needed to assure that the number $j = (l_0)^{i_0}$ appearing in the proof is an element of log^k. It is quite possible that this assumption is not optimal; we have not made a serious effort to improve it.

¿From the proof of theorem 5.1 we obtain the following corollary:

Corollary 5.2 *Assume $K > k+1$. Then $Sk(\Sigma_n^b, length\ log^{k-2}) \vdash Sk(\Sigma_{n+1}^b, length\ log^k)$.*

Proof. The corollary follows immediately from the following observation.

Let $\hat{\Lambda}, l_0, j$ be as in the proof of theorem 5.1. Then there is a Λ with the properties required in the proof of theorem 5.1 such that $lh(\Lambda) \in \log^{k-2}$.

Let us prove this observation.

For $i \leq j$, let L_i denote the number of terms of depth at most i which have to be included in Λ. Then $L_0 \leq l_0 + 1$ and $L_{i+1} \leq L_i + L_i^{N-1} \cdot N \leq L_i^{N+1}$. Hence $L_j \leq (l_0+1)^{(N+1)^j}$. Since $j \in \log^k$, $(N+1)^j \in \log^{k-1}$ and $L_j \in \log^{k-2}$. □

There is no direct connection between $Sk(\Sigma_n^b, depth\ log^k)$ and $Sk(\Sigma_n^b, length\ log^{k-2})$.

Based on the above results, we may summarize the known relationships between the theories $I\Sigma_n^b|log^k$ and $Sk(\Sigma_n^b, length\ log^k)$, for various n and k, in the following diagram:

$$Sk(\Sigma_n^b, length\ log^k) \Longleftarrow I\Sigma_{n+1}^b|log^k \longleftarrow Sk(\Sigma_{n+1}^b, length\ log^k)$$

$$\Downarrow (*)$$

$$\Uparrow \qquad I\Sigma_n^b|log^{k-1} \qquad \Uparrow$$

$$\Uparrow$$

$$Sk(\Sigma_{n-1}^b, length\ log^{k-2}) \Longleftarrow I\Sigma_n^b|log^{k-2} \longleftarrow Sk(\Sigma_n^b, length\ log^{k-2})$$

It is assumed in the diagram that $k \geq 3$ and that $n \geq n_0 + k$, $K > k+1$ (the dependence of n on k is to have T_0 implied by all the theories in question). Thick arrows denote provability, thin arrows denote inducing a model in the sense of theorem 4.3. The arrow marked with an asterisk is the one whose reversibility is a famous open problem.

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