Delocalization of two-dimensional random surfaces

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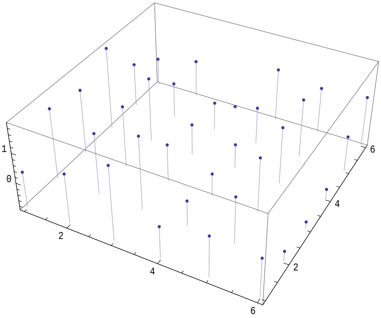
joint with Ron Peled and Maxime Gagnebin

Będlewo - 16.05.2017

Outline

- Definition of the model
- Statement of the results
- Remarks on the proof methods
- Background and open questions
- Relations with branching processes
- Outline of the proof (pictures :))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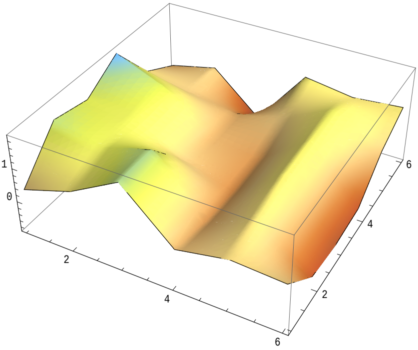
1
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Model

Definition (Surface model)

Formally μ_n can be viewed as

$$\mu_n(\mathrm{d}\varphi) \sim \exp\Big(-\sum_{i\sim j} U(\varphi_i - \varphi_j)\Big) \prod_{i\in\Lambda_n\setminus\{(n,n)\}} \mathrm{d}\varphi_i \;\; \delta_0(\varphi_{(n,n)}),$$

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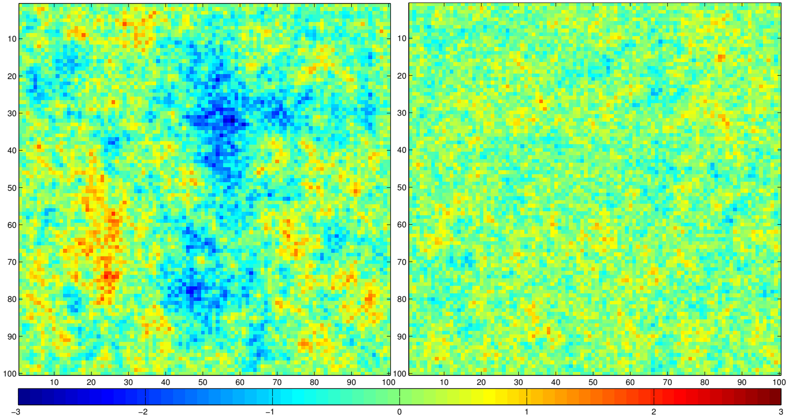
One expects that the large scale behavior of such surfaces "does not depend" on the details of the potential U.
Our assumptions:

$$\int_{\mathbb{R}} e^{-(2+\epsilon)U(x)} \mathrm{d}x < +\infty, \quad \int_{\mathbb{R}} e^{-(2-\epsilon)U(x)} \mathrm{d}x < +\infty.$$


10
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Result

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There exists a constant $C_1 > 0$ such that for all n,

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Theorem (delocalisation - probability)

For any $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mathbb{P}_{\mu_n}(|\varphi_{(0,0)}| \geq \delta\sqrt{\log n}) \geq (1-\epsilon).$$

Result detailed versions

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Other, more detailed, results are also available.

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- The proof is a version of the Mermin-Wagner argument.
- An important problem is lack of smoothness.
- Our implementation adapts methods developed by Richthammer 2007.
- Reflection positivity techniques are used to show that the the “bad regions of potential” are sparse.

Background

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- Results for higher dimensional models and discrete analogues.

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- Discrete version of the model.
- Universality!!!

Thank you for your attention!