HORN'S PROBLEM, AND FOURIER ANALYSIS

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Let A and B be two $n \times n$ Hermitian matrices. Assume that the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are knowm, as well as the eigenvalues $\beta_1, \ldots, \beta_n$ of B. What can be said about the eigenvalues of the sum $C = A + B$? This is Horn's problem. In 1962 Horn proposed a conjecture, the so-called Horn's conjecture, which says: the set of possible eigenvalues $\gamma_1, \ldots, \gamma_n$ for C is determined by a system of linear inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

where (I, J, K) is a triple of subsets of $\{1, 2, \ldots, n\}$ which is *admissible* (in a sense to be given). Horn's conjecture has been proven by Klyachko in 1998.

We revisit this problem from a probabilistic point of view. The set of Hermitian matrices X with spectrum $\{\alpha_1, \ldots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$: $X \mapsto UXU^*$ $(U \in U(n))$. Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$, and the random Hermitian matrix Y is uniformly distributed on $\mathcal{O}_\beta$. In this talk we will present a formula for the joint distribution of the eigenvalues of the sum $Z = X + Y$. The proof involves orbital measures with their Fourier transforms, and Heckman's measures.