PRECISE LARGE DEVIATIONS ASYMPTOTICS FOR PRODUCTS OF RANDOM MATRICES

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Let $S_n = \sum_{i=1}^n X_i$ be a sum of independent non-lattice random variables $(X_i)_{i\geq 1}$ under assumption $\Lambda(s) = \log \mathbb{E}[e^{sX_1}] < +\infty$, for some $s > 0$, and let $q = \Lambda'(s)$. Denote by Λ^* the Frenchel-Legendre transformation of Λ. Bahadur and Rao [Ann. Math. Stat., **31**(1960), 1015-1027] and Petrov [Theory. Prob. Appl., **10**(1965), 287-298] have established exact large deviation expansions of the followin form $\mathbb{P}(S_n \geq n(q+l_n)) \sim \frac{c(s)}{\sqrt{n}} \exp(-n\Lambda^*(s+l_n))$ as $l_n \to 0$ and $n \to \infty$. These milestone results have numerous applications in a variety of problems in pure and applied probability.

The goal is to prove equivalent expansions for the products of random matrices. Specifically, consider the product $G_n := g_n \cdots g_1$, where $(g_n)_{n\geq 1}$ is a sequence of i.i.d. $d \times d$ real random matrices of the same law μ. Assume that the support of μ is strongly irreducible and proximal for invertible matrices or allowable, contains at least one strictly positive matrix and is non-arithmetic for positive matrices. Let $I_\mu = \{s \geq 0 : \mathbb{E}(\|g_1\|^s) < +\infty\}$. Denote $\kappa(s) = \lim_{n\to\infty} (\mathbb{E}\|G_n\|^s)^{\frac{1}{n}}$ and $\Lambda(s) = \log \kappa(s)$, $s \in I_\mu$. We prove large deviation expansions for the probability $\mathbb{P}(\log |G_n x| \geq n(q + l_n))$ as $n \to \infty$, where x is a starting point on the unit sphere, $q = \Lambda'(s)$ and $l_n \to 0$. The asymptotics are expressed in terms of the eigenfunctions and invariant measures of the transfer operators P^s related to the Markov chain representation of $\log |G_n x|$ and $\log G_n^{i,j}$.

A typical result is as follows. Denote by r_s the strictly positive eigenfunction of the transfer operator P^s corresponding to the eigenvalue $\kappa(s)$ and by π_s the unique invariant probability measure of the normalized transfer operator P^s. Set also $\sigma_s^2 = \Lambda''(s)$, which, under the adopted assumptions, is a positive number.

Theorem 1. *Let $s \in I_\mu^\circ$ and $q = \Lambda'(s)$. Under appropriate moment assumptions, for any positive sequence $(l_n)_{n\geq 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, uniformly in x on the unit sphere and $|l| \leq l_n$,*

$$\mathbb{P}(\log |G_n x| \geq n(q+l)) \sim \frac{r_s(x)\pi_s(r_s^{-1})}{s\sigma_s\sqrt{2\pi n}} \exp\left(-n\Lambda^*(q+l)\right) \quad \textit{as} \quad n \to \infty.$$

Moreover, the rate function $\Lambda^*(q+l)$ admits the following expansion: for l in a small neighborhood of 0,

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s\Big(\frac{l}{\sigma_s}\Big),$$

where $\zeta_s(t)$ is the Cramér series $\zeta_s(t) = \sum_{k=3}^\infty c_{s,k} t^{k-3} = \frac{m_{3,s}}{6\sigma_s^3} + O(t)$, $m_{3,s} = \Lambda^{(3)}(s)$.

A similar large deviation expansion is established for the entries of positive matrices, under the Kesten condition, where, for the proofs, we develop a spectral gap theory for a cocycle related the scalar product. These results are based on a joint work with Hui Xiao and Quansheng Liu.