ON PERPETUITIES WITH LIGHT TAILS

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We consider a random variable R defined as a solution of the affine stochastic equation

$$R \stackrel{d}{=} MR + Q \qquad R \text{ and } (M, Q) \text{ independent.}$$

Under suitable conditions, R may be represented as the series

$$R \stackrel{d}{=} \sum_{j=1}^{\infty} Q_j \prod_{k=1}^{j-1} M_k, \qquad (M_n, Q_n)_{n=1}^{\infty} \text{ are i.i.d. copies of } (M, Q).$$

We examine the asymptotics of logarithmic tails of a perpetuity R in the case when $\mathbb{P}(M \in [0, 1)) = 1$ and Q has all exponential moments. If M and Q are independent, under regular variation assumptions, we show that

$$\lim_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} = -c,$$

where constant $c > 0$ is given explicitly and

$$h(x) = \inf_{t\geq 1} \left\{ -t \log \mathbb{P}\left(\frac{1}{1-M} > t, Q > \frac{x}{t} \right) \right\}.$$

Moreover, we deal with the case of dependent M and Q and give asymptotic bounds for $-\log \mathbb{P}(R > x)$. It turns out that dependence structure between M and Q has a significant impact on the asymptotic rate of logarithmic tails of R. Such phenomenon is not observed in the case of heavy-tailed perpetuities. We show that

$$-c_1 \leq \liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{co}(x)} \leq -c_2,$$

where c_1 and c_2 are positive and explicit and

$$h_{co}(x) = \inf_{t\geq 1} \left\{ -t \log \min \left\{ \mathbb{P}\left(\frac{1}{1-M} > t \right), \mathbb{P}\left(Q > \frac{x}{t} \right) \right\} \right\}$$

is the h function corresponding to comonotonic (M, Q). If the vector (M, Q) is positively quadrant dependent, then we show that

$$-d_1 \leq \liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{ind}(x)} \leq \limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{ind}(x)} \leq -d_2,$$

where d_1 and d_2 are explicit and optimal and

$$h_{ind}(x) = \inf_{t\geq 1} \left\{ -t \log \mathbb{P}\left(\frac{1}{1-M} > t \right) - t \log \mathbb{P}\left(Q > \frac{x}{t} \right) \right\}$$