SHARP MULTIPLIER THEOREM FOR HARDY SPACES ON SPACES OF HOMOGENEOUS TYPE

EDYTA KANIA

The classical Hörmader multiplier theorem states that if m satisfies

$$\text{(S)} \qquad \sup_{t>0} ||\eta(\cdot)m(t\cdot)||_{W^{2,\beta}(\mathbb{R})} \leq c_\eta,$$

with some $\beta > d/2$, then operator $m(-\Delta)$ is of weak type (1,1) and bounded on $L^p(\mathbb{R}^d)$ for every $p \in (1, \infty)$. Here $0 \leq \eta \in C_c^\infty(2^{-1}, 2)$ is fixed and $W^{2,\beta}(\mathbb{R})$ is the standard L^2-Sobolev norm on $\mathbb{R}$. It is well-known that the constant $d/2$ is sharp. Similar multiplier theorem is known on the Hardy space $H^1(\mathbb{R}^d)$. In the talk we shall consider some generalizations of this results.

Suppose the metric-measure space (X, ρ, μ) satisfies the doubling condition, which implies that there exists constant d such that

$$\text{(D)} \qquad \mu(B(x, \gamma r)) \leq C(1+\gamma)^d \mu(B(x, r)), \qquad \gamma > 0,\, x \in X,\, r > 0.$$

Assume that the semigroup $P_t = \exp(-tA)$ generated by a self-adjoint positive operator A satisfies the lower and upper gaussian bounds. Moreover, suppose that there exists $C > 0$, such that for every $R > 0$ and a measurable function m on $\mathbb{R}$ such that $\operatorname{supp}(m) \subseteq [R/2, R]$ we have

$$\text{(P)} \qquad \int_X |K_{m(A)}(x, y)|^2 d\mu(x) \leq C \mu(B(y, R^{-1/2}))^{-1} ||m(R\cdot)||^2_{L^2(X)}.$$

We prove that if the bounded function m satisfies (S) with $\beta > d/2$, then $m(A)$ is bounded from $H^1(A)$ to $H^1(A)$, where $H^1(A) = \{f \in L^1(X) : \| \sup_{t>0} |P_t f| \|_{L^1(X,\mu)} < \infty\}$ denotes the Hardy space associated with A.

As a main example we consider the multiparameter Bessel operator $B = B_1 + ... + B_N$, where $B_i f(x) = -\delta_j^2 f(x) - \alpha_j/x_j \delta_j f(x)$, $\alpha_j > -1$, on the space $(0, \infty)^N$ with the Euclidean metric and the measure $x^{\alpha_1} ... x_N^{\alpha_N} dx_1 ... dx_N$. The condition (D) is satisfied with $d = \sum_{j=1}^N \max(1, 1+\alpha_j)$. By analysing the imaginary powers $B^{ib}, b \in \mathbb{R}$, of the one-dimensional Bessel operator we show that in this case the constant $d/2$ is sharp.

The talk is based on the joint work with Marcin Preisner (University of Wrocław).