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**Summability and global property of transseries solution of Hamiltonian system**

Let $H_0$ and $H_1$ be given, respectively, by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{1}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \tag{2}$$

where $B_j(q_1, s, q)$'s are holomorphic at the origin, $\lambda_j$'s are constants and $\sigma$ is a positive integer. Define $H := H_0 + H_1$. Consider the Hamitonian system with $n$ degrees of freedom

$$\dot{q}_j = \nabla_{p_j} H, \quad \dot{p}_j = -\nabla_{q_j} H, \quad j = 1, 2, \ldots, n. \tag{3}$$

Assume

$$B_\nu \equiv B_\nu(q_1, q_1^{2\sigma} p_1, q) = B_{\nu,0}(q_1, q) + q_1^{2\sigma} p_1 B_{\nu,1}(q_1, q), \quad \nu = 2, \ldots, n, \tag{4}$$

where $B_{\nu,0}$ and $B_{\nu,1}$ are analytic at $(q_1, q) = (0, 0)$. Suppose that the Poincaré condition holds

$$Re\, \lambda_j > 0, \quad j = 2, 3, \ldots, n. \tag{5}$$

Assume the nonresonance condition

$$\sum_{\nu=2}^{n} \lambda_\nu k_\nu - \lambda_j \neq 0, \quad \forall\ k_\nu \in \mathbf{Z}_+,\ \nu = 2, \ldots, n,\ j = 2, \ldots, n. \tag{6}$$

Set $\lambda = (\lambda_2, \ldots, \lambda_n)$. The formal transseries is defined by

$$\sum_{k \geq k_0,\ \ell \geq \ell_0} c_{k,\ell}\, t^{-\frac{\ell}{2\sigma-1}} e^{\lambda k t}, \tag{7}$$

where $k = (k_2, \ldots, k_n)$, $\lambda k = \lambda_2 k_2 + \cdots + \lambda_n k_n$. Here $c_{k,\ell}$'s are complex constants, $k_0$ is a multiinteger and $\ell_0 \geq 0$ is an integer. Consider the formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$. Then we have

**Proposition** Suppose that (4),(5) and (6) are satisfied. Then there exists a formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ of (3) in the domain $\{t | Re(\lambda_j t) < 0, j = 2, \ldots, n\}$.

**Theorem** Suppose that (4), (5) and (6) are satisfied. Then the formal transseries solution $(q_1(t), p_1(t), q(t), p(t))$ is $(2\sigma - 1)$-Borel summable in every direction in $\{t | Re(\lambda_j t) < 0, j = 2, \ldots, n\}$. There exists a neighborhood of $t = 0$, $\Omega_1$ such that the Borel sum is the analytic transseries solution of (3) in the domain $\{t | Re(\lambda_j t) < 0, j = 2, \ldots, n\} \cap \Omega_1$.

**Example**. The following Hamiltonian is the local counterpart of the Hamiltonian studied by Taimanov related with the nonintegrability of a geodesic flow

$$H_1 := c q_1^{4\sigma} p_1^2 + \sum_{j=2}^{n} B_j(q_1) p_j^2, \tag{8}$$

where $c$ is a constant and $B_j(q_1)$ is an analytic function in some neighborhood of $q_1 = 0$. For $H_0$ in (1), we define $H := H_0 + H_1$. We see that $\chi_H$ is not $C^\omega$-Liouville integrable at the origin under a certain condition, while it is $C^\infty$-Liouville integrable at the origin.

**Analytic continuation and connection problem of transseries solution**.

We discuss the analytic continuation of the summed transseries solution. We use first integrals in order to study the global property of the summed transseries solution. The idea is closely related with the notion of a semi-formal solution defined by Balser. (cf. [1]).

## References

[1] Balser, W. Semi-formal theory and Stokes' phenomenon of nonlinear meromorphic systems of ordinary differential equations, Banach Center Publications 97 (2012), 11-28.

[2] Costin, O. Topological construction of transseries and introduction to generalized Borel summability, Contemporary Mathematics. 2005; 373: 137-75.

[3] Kuik, K. Transseries in differential and difference equations, PhD Thesis, University of Groningen. 2003; ISBN 90-367-1771-x.