Homogeneous isosceles-free spaces

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joint work with

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Homogeneity properties of metric spaces

A metric space X is said to be

- (uniquely) n-homogeneous for $n \in \mathbb{N}_+$ if for every isometry $f\colon A \to B$ between subspaces $A, B \subseteq X$ with $(0 <)|A| \leq n$ there exists a (unique) automorphism $F\colon X \to X$ extending f, i.e. $F|_A = f$;
- (uniquely) ultrahomogeneous if it is (uniquely) n-homogeneous for every $n \in \mathbb{N}_+$.

Note that X is uniquely n-homogeneous if and only if it is n-homogeneous and uniquely 1-homogeneous.

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Isosceles-free spaces

A metric space (X, d) is called isosceles-free if $d(x, y) \neq d(x, z)$ for distinct points $x, y, z \in X$.

Example

Introduced by Janoš and Martin in 1978 under the name "star rigid".
An isosceles-free space is zero dimensional Hattori (1990).

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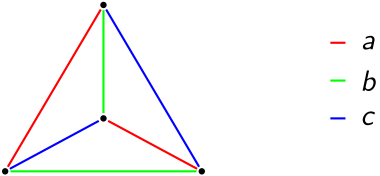

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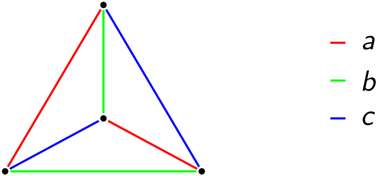

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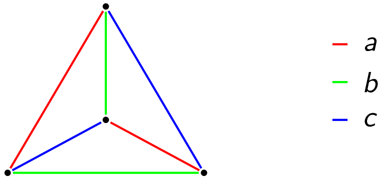

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Isosceles-free spaces and embeddings

Proposition

Let X be a metric space and Y is isosceles-free. For every $x \in X$, $y \in Y$ there exists at most one isometric embedding $f : X \to Y$ with $f(x) = y$.

Proof.

Let f, g be isometric embeddings with $f(x) = y = g(x)$. For $x' \in X$, we have $d(y, f(x')) = d(x, x') = d(y, g(x'))$
$\Rightarrow f(x') = g(x')$. □

Corollary

Every 1*-homogeneous isosceles-free space X is uniquely* 1*-homogeneous.*

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Proposition

Every 1-homogeneous isosceles-free space X is ultrahomogeneous.

Proof.

Let $A, B \subset X$ finite, $i\colon A \to B$ isometry. For fixed $x \in A$ exists automorphism $f\colon X \to X$ with $f(x) = i(x)$.Then $f|_A$ and i are isometric embeddings $A \to X$ with $x \mapsto i(x)$, hence $f|_A = i$ by the previous Proposition. □

Hence the notation homogeneous isosceles free space makes sense.

Proposition

A metric space X is homogeneous isosceles-free if and only if it is uniquely 2-homogeneous.

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The automorphism group of isosceles-free spaces

Recall: A group G is Boolean $g^2 = 1$ for all $g \in G$. Note that Boolean groups are Abelian.

Proposition

For every isosceles-free space X the isometry group $\mathrm{Aut}(X)$ is Boolean.

We consider $\mathrm{Dist}(X) = \{d(x,y) \colon x, y \in X\}$ and for $a \in X$

- the *distance map* $D_a \colon X \to \mathrm{Dist}(X)$, $x \mapsto d(x,a)$,
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Observe that X is homogeneous isosceles-free iff D_a and E_a are bijective for every $a \in X$.

Corollary

For every finite homogeneous isosceles-free metric space X we have $|X| = 2^m$ for some $m \in \omega$.

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For every finite homogeneous isosceles-free metric space X we have $|X| = 2^m$ for some $m \in \omega$.

We consider the class $\mathcal{K}$ of finite isosceles-free spaces (with isometric embeddings).

Does it have the amalgamation property? No

What about the weak amalgamation property?

Theorem

The class of all finite isosceles-free spaces does not have the weak amalgamation property.

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$$
\begin{array}{ccccc}
 & & & X & \\
 & & \overset{\psi_X}{\nearrow} & & \overset{\pi_X}{\searrow} \\
A & \xrightarrow{\phi} & B & & & Z \\
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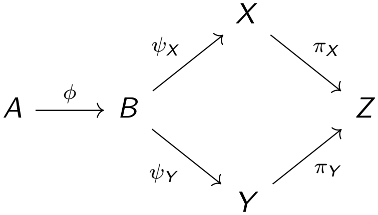

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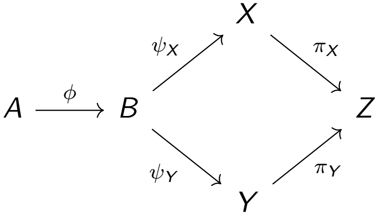

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Normed $\mathbb{Z}_2$-linear spaces and Boolean metric spaces

By a norm on an Abelian group X we mean a map $\|\cdot\|\colon X \to [0,\infty)$ such that

1. $\|x\| = 0$ if and only if $x = 0$, for $x \in X$,
2. $\|x + y\| \leq \|x\| + \|y\|$, for $x, y \in X$,
3. $\|-x\| = \|x\|$, for $x \in X$.

If X is even Boolean, i.e. $x = -x$, then X can be interpreted as $\mathbb{Z}_2$-linear space and every norm satisfies $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for every $\alpha \in \mathbb{Z}_2$ and $x \in X$, so it is a $\mathbb{Z}_2$-norm.

We call a metric space Y Boolean if $\mathrm{Aut}(Y)$ is a Boolean group.

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We call a metric space Y Boolean if $\mathrm{Aut}(Y)$ is a Boolean group.

Proposition

Let X be a 1-homogeneous space such that $\mathrm{Aut}(X)$ is Abelian.

1. *For $f \in \mathrm{Aut}(X)$, $d(x, f(x))$ does not depend on $x \in X$.*
2. *$\|f\| := d(x, f(x))$ for $x \in X$ defines a norm on $\langle \mathrm{Aut}(X), \circ \rangle$.*
3. *X is uniquely 1-homogeneous.*
4. *$E_a \colon \mathrm{Aut}(X) \to X, f \mapsto f(a)$ is an isometry for every $a \in X$.*
5. *$\mathrm{Aut}(X)$ is a Boolean group.*

Corollary

For a 1-homogeneous metric space X, $\mathrm{Aut}(X)$ is Abelian iff it is Boolean. In this case, X is uniquely 1-homogeneous.

1-homogeneous spaces and norms on Aut(X)

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Let X be a 1-homogeneous space such that Aut(X) *is Abelian.*

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Let X be a Boolean metric space. $\mathrm{Aut}(X)$ *is a normed* $\mathbb{Z}_2$*-linear space, and the canonical action of* $\mathrm{Aut}(X)$ *on X turns X into an affine space over* $\mathrm{Aut}(X)$*. Moreover, every evaluation map* $E_a\colon \mathrm{Aut}(X) \to X$ *is an affine isometry.*

Moreover, for a metric space X we have the following.

1. X is Boolean if and only if it is isometric to $\mathbb{Z}_2$-normed space and uniquely 1-homogeneous.
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Representation of homogenous isosceles-free spaces

Given a $\mathbb{Z}_2$-linear space, choosing a basis we obtain an isomorphism to $\mathbb{Z}_2^{(I)} = \{v \in \mathbb{Z}_2^I \colon v \text{ has finite support}\}$.

We view $\mathbb{Z}_2^{(I)}$ as the family $\mathcal{P}_\omega(I)$ of all finite subsets of I with the operation of symmetric difference: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. We use the notation $2^{(I)}$ for $\langle \mathcal{P}_\omega(I), \triangle \rangle$.

Theorem

Let I be a set and let $\|\cdot\| \colon 2^{(I)} \to [0, \infty)$ be an injective map satisfying $\|\emptyset\| = 0$ and $\|x \triangle y\| \leq \|x\| + \|y\|$ for $x, y \in 2^{(I)}$. By putting $d(x, y) := \|x \triangle y\|$ for $x, y \in 2^{(I)}$ we obtain a homogeneous isosceles-free space. Moreover, every homogeneous isosceles-free space can be obtained this way up to an isometry.

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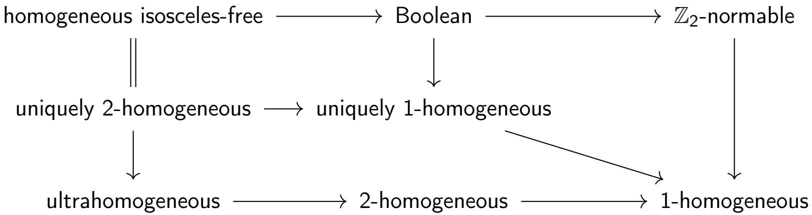
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Decomposition in isosceles free components

We call $S_X := \{r \in \mathrm{Dist}(X) \colon \forall x \in X \exists! y \in X \text{ with } d(x,y) = r\}$ the set of singleton distances.

We call an equivalence relation $\sim$ on X invariant if for every $f \in \mathrm{Aut}(X)$ we have $f(x) \sim f(y)$ iff $x \sim y$.

Theorem

Let X be a 2-homogeneous space and let $x \sim y$ if $d(x,y) \in S_X$ for $x, y \in X$. Then, $\sim$ is an invariant equivalence relation inducing a decomposition of X into pairwise isometric homogeneous isosceles-free spaces.

Example: Decomposition of C_4 (blackboard).

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Let X be a 2-homogeneous space and let $x \sim y$ if $d(x,y) \in S_X$ for $x, y \in X$. Then, $\sim$ is an invariant equivalence relation inducing a decomposition of X into pairwise isometric homogeneous isosceles-free spaces.

Example: Decomposition of C_4 (blackboard).

A somewhat dual construction: isosceles-generated components

Theorem

Let X be a 1-homogeneous space. Let $x \sim y$ if there is $z \neq y$ such that $d(x,y) = d(x,z)$, i.e. we collapse all non-degenerate isosceles triangles.

1. *$\sim$ is invariant and induced a decomposition into isometric 1-homogeneous components. In particular, automorphisms map components onto components.*
2. *f $|X/{\sim}| \geq 2$, then X is uniquely 1-homogeneous.*
3. *Every $f \in \mathrm{Aut}(X)$ either fixes all components C (setwise), or none of them. In the latter case we have $f \circ f = \mathrm{id}$.*
4. *If $|X/{\sim}| \geq 3$, then $\mathrm{Aut}(X)$ is Boolean, i.e. X is a Boolean metric space.*

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A metric space X is called isosceles-generated if its decomposition into isosceles-generated components has at most one component.

In other words, we have two extreme cases:

1. All isosceles-generated components of X are singletons, then X is isosceles-free.
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The Rainbow Duplicate

Let X be a 1-homogeneous space, $H \leq \mathrm{Aut}(X)$ an Abelian subgroup s. t. for every $x, y \in X$ there is a unique element $h \in H$ (denoted by h_x^y) s. t. $h(x) = y$. We define the rainbow duplicate of X as the metric space $X \times_r 2$ with the distance

$$d(\langle x, 0\rangle, \langle y, 0\rangle) = d(\langle x, 1\rangle, \langle y, 1\rangle) = d_X(x, y),$$
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where $r \colon H \to (0, \infty) \setminus \mathrm{Dist}(X)$ is an injective map s. t. triangle inequality in $X \times_r 2$ is satisfied. We also suppose there exists a map $g \in \mathrm{Aut}(X)$ s. t. $g \circ g = \mathrm{id}$ and $g \circ h \circ g^{-1} = h^{-1}$ for every $h \in H$.

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Properties of the rainbow duplicate

X is 1-homogeneous, $H \leq \mathrm{Aut}(X)$ an Abelian with unique $h_x^y \in H$ with $h_x^y(x) = y$ for $x, y \in X$ and $r\colon H \to (0, \infty) \setminus \mathrm{Dist}(X)$ injective.

Properties of $X \times_r 2$

1. $|r(h) - r(h')| \leq \min(\mathrm{Dist}(X) \setminus \{0\})$ and $r(h) \geq \max(\mathrm{Dist}(X))$ for every $h, h' \in H$, imply the triangle inequality of d.
2. $\mathrm{Dist}(X \times_r 2) = \mathrm{Dist}(X) \cup \mathrm{im}(r)$, which is a disjoint union.
3. The decomposition of $X \times_r 2$ into isosceles-generated components refines $\{X \times \{0\}, X \times \{1\}\}$, with equality iff X is isosceles-generated.
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General 1-homogeneous metric spaces

Theorem

Let X be a 1-homogeneous metric space. Then one of the following is true.

1. *X is isosceles-generated.*
2. *X is a rainbow duplicate of an isosceles-generated space.*
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Suppose that X is even 2-homogeneous. Then one of the following is true.

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Maximal number of distances in homogeneous spaces

Let X be a metric space with n elements.
Two simple observations:

1. If X has the discrete metric ($\delta(X) := |\operatorname{Dist}(X)| = 2$), it is ultrahomoegenous.
2. If X is 1-homogeneous, then $\delta(X) \leq n$.

Question: What is the maximal number of distances, so that X can be k-homogeneous/ultrahomogeneous?

For $k \in \mathbb{N}$ we set

$$\Delta_k(n) := \max\{\delta(X) : X \text{ a } k\text{-homogeneous space with } |X| = n\},$$
$$\Delta_\omega(n) := \max\{\delta(X) : X \text{ an ultrahomogeneous space with } |X| = n\}.$$

Clearly, we have $\Delta_\omega(n) \leq \cdots \leq \Delta_2(n) \leq \Delta_1(n)$ for every $n \in \mathbb{N}_+$.

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A first upper bound for 1-homogeneous spaces

Let X be a finite 1-homogeneous n-point space. Clearly, $\delta(X) \leq n$. Recall that S_X is the set of "singleton distances". We have $\delta(X) \leq (|S_X| + |X|)/2$, because

$$D_x \colon X \to \mathrm{Dist}(X), \qquad y \mapsto d(x, y)$$

is a surjection where exactly the elements of S_X have a unique preimage. Hence, $|X| \geq |S_X| + 2(\delta(X) - |S_X|)$.

Proposition

We have $\delta(X) = n$ if and only if X is isosceles-free, and in this case X is ultrahomogeneous and n is a power of two. In other words, $\Delta_\omega(n) = \Delta_1(n) = n$ if $n = 2^m$, and $\Delta_1(n) < n$ otherwise.

For $n = 2^m$ the bound is optimal, since X_n is a homogenous isosceles-free space with n elements.

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An upper bound for 2-homogeneous spaces

Theorem

For a finite 2-homogeneous metric space X of $n := |X|$ elements, where $n = 2^m(2k+1)$, we have $\delta(X) \leq 2^m(k+1) =: \beta_n$.

Proof.

Consider the decomposition of X into isosceles-free components $X/\sim$. We have $|S_X| = |C|$ for any $C \in X/\sim$. Moreover, $|C| = 2^p$ for some p since C is homogenous isosceles-free, and $|C| \leq 2^m$ since $X/\sim$ is a decomposition into pairwise-isometric subspaces. Hence,

$$\delta(X) \leq \frac{|S_X| + |X|}{2} \leq \frac{2^m + 2^m(2k+1)}{2} = 2^m(k+1). \qquad \square$$

The space $B_{m,k} = 2^m C_{2k+1} \times_1 \langle 2^m, \|\cdot\|\rangle$ shows optimality.

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Corollary

For every $n \in \mathbb{N}_+$ we have $\Delta_\omega(n) = \Delta_2(n) = \beta_n \leq \Delta_1(n) \leq n$.

What more can we say about 1-homogeneous spaces?
Let X be a 1-homogeneous metric space X with n elements.

Proposition

If $n = 2k+1$, then $\delta(X) \leq k+1$. Hence, $\Delta_1(n) = \Delta_2(n) = \beta_n$ for every odd n.

Main ingredient in the proof: $S_X = \{0\}$ if n is odd.
The space $D_{2n} = C_n \times_r 2$ has $2n$ elements and $\delta(D_{2n}) > \beta_{2n}$.

Corollary

We have $\Delta_2(n) < \Delta_1(n) < n$ for even n that is not a power of two.

More on 1-homogeneous spaces

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For every $n \in \mathbb{N}_+$ we have $\Delta_\omega(n) = \Delta_2(n) = \beta_n \leq \Delta_1(n) \leq n$.

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The number of distances in a 1-homogeneous space X of cardinality $n = 2(2k+1)$ with $2k+1$ prime is bounded from above by $3k+2$. Hence, $\Delta_1(n) = \alpha_n$ for such n.

We have an example showing that this bound is optimal too.

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We have $\Delta_1(n) \leq n-2$ for every $n \geq 7$ that is not a power of two.

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We have $\Delta_1(n) \leq n-2$ *for every* $n \geq 7$ *that is not a power of two.*

n	$\Delta_2(n)$	$\Delta_1(n)$	argument for Δ_1 upper bound
1	1	1	power of two
2	2	2	power of two
3	2	2	odd
4	4	4	power of two
5	3	3	odd
6	4	5	two times odd prime
7	4	4	odd
8	8	8	power of two
9	5	5	odd
10	6	8	two times odd prime
11	6	6	odd
12	8	10	$\Delta_1(n) \leq n - 2$
13	7	7	odd
14	8	11	two times odd prime
15	8	8	odd
16	16	16	power of two
17	9	9	odd
18	10	$\geq 14, \leq 16$	$\Delta_1(n) \leq n - 2$
19	10	10	odd
20	12	$\geq 16, \leq 18$	$\Delta_1(n) \leq n - 2$

The End

Thank you for your attention!

References

C. Bargetz, A. Bartoš, W. Kubiś, F. Luggin: Homogeneous isosceles-free spaces. Preprint (arXiv:2305.03163)