Constructing Compacta From Posets

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Posets vs Compacta

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- Various dualities exist between certain posets and compacta:

Boolean Algebras	$\leftrightarrow$	T_2 0-Dim Compacta	(Stone 1936)
Separative Lattices	$\leftrightarrow$	T_1 Compacta	(Wallman 1938)
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- More promising avenue for constructions – trees or tree-like posets as considered in set theory and topological dynamics.

Trees and their Spectra

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- Then $\mathsf{S}\mathbb{P}$ is a 0-dimensional metrisable compactum.

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- Take a 0-dimensional metric compactum X.
- Let $\mathbb{P}_0$ be a clopen (finite) partition of X.
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$$\mathbb{P} = \bigcup_{k \in \omega} \mathbb{P}_k \qquad \text{with } n^{\mathrm{th}} \text{ cone} \qquad \mathbb{P}^n = \bigcup_{k=0}^{n} \mathbb{P}_k.$$

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- Points $x \in X$ correspond to branches $S_x = \{p \in \mathbb{P} : x \in p\}$.
- Moreover, $x \mapsto S_x$ is a homeomorphism from X onto $\mathsf{S}\mathbb{P}$. $\square$

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- In fact this characterises $C \subseteq \mathbb{P}$ forming covers $(p^{\in})_{p\in C}$ of $\mathrm{S}\mathbb{P}$.

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- Suggests defining 'branches' to select elements from 'levels'.

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- C is refined by a band $B \subseteq \mathbb{P}$. Suffices to show $X = \bigcup B$.

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- But also $y \in q \setminus b$ and hence $q \nsubseteq b$, a contradiction. □

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- The $\mathbb{P}$ above turns out to be quite 'tree-like'...

ω-Posets

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Proposition (Bartoš-B.-Vignati 2023)

If $\mathbb{P} = \{p_n : n \in \omega\}$ is a basis of a metric compactum X then

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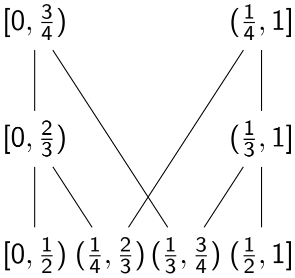

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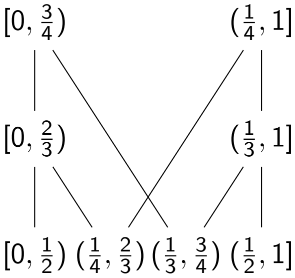

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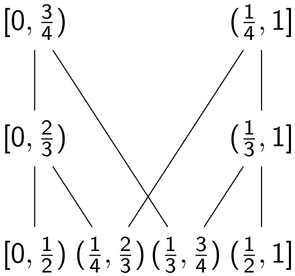

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Theorem (Bartoš-B.-Vignati 2023)

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If $\mathbb{P}$ is an ω-cap-basis of a T_1 compactum X then

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- In this way we obtain a duality between appropriate categories of ω-posets and metrisable compacta.

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Theorem (Bartoš-B.-Vignati 2023)

If $\mathbb{P}$ is a regular ω-poset then $S\mathbb{P}$ is a regular compactum.

- Converse also holds under a mild 'primeness' condition on $\mathbb{P}$.

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discrete graphs with surjective functions	$\rightarrow$	Cantor space
path graphs with monotone relations	$\rightarrow$	unit interval
path graphs with surjective relations	$\rightarrow$	pseudoarc
fan graphs with spoke-monotone relations	$\rightarrow$	the Lelek fan
connected graphs with monotone relations	$\rightarrow$	the Menger curve

Building Graded ω-Posets

- Graded posets can be built from relations between finite sets, e.g. edge-preserving relations between finite graphs.
- Edges in the graph then correspond to overlaps of open sets.
- The graphs thus specify the 'shape' of the open covers.
- This can be formalised as forming posets from sequences of relational morphisms in specific subcategories of graphs.

subcategory $\rightarrow$ Fraïssé sequence $\rightarrow$ ω-poset $\rightarrow$ compactum

discrete graphs with surjective functions	$\rightarrow$	Cantor space
path graphs with monotone relations	$\rightarrow$	unit interval
path graphs with surjective relations	$\rightarrow$	pseudoarc
fan graphs with spoke-monotone relations	$\rightarrow$	the Lelek fan
connected graphs with monotone relations	$\rightarrow$	the Menger curve

- Like (Irwin-Solecki 2006) and (Debski-Tymchatyn 2018) but they only consider functional morphisms. To obtain the desired space they also have to identify points of a 'pre-space'.