Efficiently extending partial automorphisms of graphs

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EPPA

Definition

Let $\mathscr{C}$ be a class of finite structures.

- Whenever $H \geq G$ in $\mathscr{C}$ are such that *every* partial automorphism of G is the restriction of an automorphism of H, H is called an *EPPA witness* for G.
- If every G in $\mathscr{C}$ has an EPPA witness in $\mathscr{C}$, say $\mathscr{C}$ *has EPPA*.

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Hence sometimes called the *Hrushovski Property*.

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One of the now standard methods to prove the existence of *generic automorphisms* of the Fraïssé limit of $\mathscr{C}$ involves proving that $\mathscr{C}$ has EPPA.

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Theorem (T. Gardiner (1976))

The finite homogeneous graphs are:

- *disjoint unions of cliques K_n, complements of these;*
- *The 5-cycle C_5;*
- *$L(K_{3,3})$, the line graph of complete bipartite graph $K_{3,3}$.*

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Exercise: Is $L(K_{3,3})$ a *minimal* EPPA witness for these graphs?

Tools via finite groups

Observation

Suppose that H is an EPPA witness for G. Then $\mathrm{Aut}(H)$ has a section (a quotient of a subgroup) isomorphic to $\mathrm{Aut}(G)$;
in particular, $|\mathrm{Aut}(G)|$ divides $|\mathrm{Aut}(H)|$.

Proof.

From the definition of EPPA witness, we see that the setwise stabiliser of $V(G)$ in $\mathrm{Aut}(H)$ induces $\mathrm{Aut}(G)$ on it. □

Tools via finite groups

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1. *H is vertex-transitive.*
2. *H is arc-transitive.*
3. *Either H is vertex-primitive, or the vertex set of G contains at most one point of any block of imprimitivity for $\mathrm{Aut}(H)$. In the latter case, the number of vertices of the EPPA witness is at least twice the number of vertices of G.*

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So minimality of an EPPA witness H can sometimes (say when $|G| < |H| < 2|G|$) can be verified by considering possibilities of primitive groups of degree $d, |G| < d < |H|$.

Scarcity of primitive permutation groups

Degree	Nr Permutation Groups $OEIS : A000019$	Nr Primitive Groups $OEIS : A000638$
1	1	1
2	1	1
3	2	2
4	4	2
5	11	5
6	19	4
7	56	7
8	96	7
9	296	11
10	554	9
11	1593	8
12	3094	6
13	10723	9
14	20832	4

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Proof.

$|L(K_{3,3})| = 9$ and $|C_6| = 6$; by Theorem 8, a smaller EPPA witness would have to have vertex-primitive automorphism group of degree 7 or 8 having the dihedral group of order $12 = \mathrm{Aut}(C_6)$ as a section.

After checking the few possibilities, see that there is no such primitive group. □

Very small EPPA witnesses

Theorem

Let G be a graph on n vertices, which has a minimum EPPA-witness H on fewer than $(5/4)n$ vertices. Then H is homogeneous.

Proof.

We say that a graph is k-homogeneous if any isomorphism between induced subgraphs on at most k vertices extends to an automorphism. We use two ingredients in the proof:

(a) (Neumann's Separation Lemma). Let A and B be subsets of the domain of a transitive permutation group G of degree n. If $|A| \cdot |B| < n$, then there exists $g \in G$ such that $Ag \cap B = \emptyset$.

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(b) (Cameron). A 5-homogeneous graph is homogeneous.

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- By (a), if A is a set of vertices of H with $|A| \leq 5$, then there exists $g \in \mathrm{Aut}(H)$ such that $Ag \cap (V(H) \setminus V(G)) = \emptyset$; in other words, $Ag \subseteq V(G)$.

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- Let A and B be subsets of $V(H)$ with $|A| \leq 5$, and $f : A \to B$ a partial isomorphism. By what we have just proved, we may assume that $A, B \subseteq V(G)$. Since H is an EPPA-witness for G, the map f extends to an automorphism of H. Thus H is 5-homogeneous.

Small EPPA witnesses

Theorem

Let G be a graph on n vertices, and H a minimum EPPA-witness for G with fewer than $2n$ vertices. Then $\mathrm{Aut}(H)$ is a rank 3 permutation group on $V(H)$.

Proof.

Repeating the above argument with 2 replacing 5, we see that H is 2-homgeneous, which means that $\mathrm{Aut}(H)$ is transitive on vertices, ordered edges, and ordered non-edges; the definition of rank 3. □

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Hints

- A rank 3 permutation group has a unique complementary pair of invariant graphs (apart from the complete and null graphs).
- If the group is imprimitive, these graphs are a disjoint union of complete graphs of the same size and its complement; these graphs are homogeneous.
- Using the Classification of Finite Simple Groups, all rank 3 permutation groups (and hence all 2-homogeneous graphs) are known (M. W. Liebeck (1987)).

Two-graphs and double covers

Now suppose the graph G has n vertices and has an EPPA witness with $2n$ vertices, whose automorphism group has n blocks of imprimitivity of size 2. We show that certain graphs G have EPPA-covers arising from two-graphs corresponding to the “switching class” of G.

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Definition

A **two-graph** is a pair (X, T), where X is a set and T a collection of 3-subsets of X such that any 4-subset of X contains an even number of members of T.

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Definition

Graphs G_1 and G_2 on the same vertex set X are **switching-equivalent** if there is a subset Y of X such that G_1 and G_2 have the same edges within our outside Y and complementary edges between Y and $X \setminus Y$.

Definition

A **double cover** of the complete graph on X is a graph on a set $\hat{X}$ with a two-to-one surjection τ to X such that

- points with the same image under τ are not adjacent;
- if $\tau(x_1) = \tau(x_2) \neq \tau(y_1) = \tau(y_2)$, there are two disjoint edges between $\{x_1, x_2\}$ and $\{y_1, y_2\}$.

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Theorem

Let (X, T) be the two-graph corresponding to a switching class S. Let D be the corresponding double cover. Then the following are equivalent:

(a) *(X, T) is homogeneous;*

(b) *D is homogeneous (as a structure with a partition into parts of size 2 in addition to the graph structure);*

(c) *D is an EPPA-witness for partial isomorphisms between graphs in S. In particular D is an EPPA-witness for any graph in S.*

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Now the natural question is: which are the homogeneous two-graphs? It turns out there are very few:

Theorem

Apart from the complete and null two-graphs, there are just two homogeneous two-graphs, on 6 and 10 points respectively.

Case $n = 6$

The double cover is the 1-skeleton of the icosahedron. There are four graphs in the switching class, falling into two complementary pairs:

- a 5-cycle with an isolated vertex (the **spokeless wheel**), and its complement, with automorphism group dihedral of order 10;
- a triangle with a pendant edge at each vertex (the **legged triangle**) and its complement, with automorphism group dihedral of order 6.

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E.g. for the spokeless wheel, the only primitive but not 2-transitive groups of degree n with $6 < n < 12$ having order divisible by 10 are S_5 and A_5 (degree 10); up to complementation, the corresponding rank 3 graph is the Petersen graph. This graph contains 5-cycles, but any point outside a 5-cycle is joined to a point in the cycle. In its complement $L(K_5)$, any vertex outside a 5-cycle is joined to two vertices of the cycle. So neither is an EPPA witness.