Lebesgue measure-preserving maps on one-dimensional manifolds

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Based on joint works with Jozef Bobok (ČVUT Prague), Piotr Oprocha (AGH Krakow & IRAFM) and Serge Troubetzkoy (Aix-Marseille).

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Univerza v Mariboru

Our papers I will mention during the talk

- Bobok, Č., Oprocha, Troubetzkoy, **Periodic points and shadowing property for generic Lebesgue measure preserving interval maps**, Nonlinearity **35** (2022), 2534-2557.

- Bobok, Č., Oprocha, Troubetzkoy, **S-limit shadowing is generic for continuous Lebesgue measure preserving circle maps**, Ergodic Th. & Dyn. Sys., **43** (1), 2023, 78–98.

- Č., Oprocha, **Parametrized families of pseudo-arc attractors: physical measures and prime ends rotations**, Proc. Lond. Math. Soc. (3), **125** (2), 2022, 318–357.

- Bobok, Č., Oprocha, Troubetzkoy, **Are generic dynamical properties stable under composition with rotations?**, arXiv:2207.07186, July 2022.

Spaces $C_\lambda(\mathbb{S}^1)$ and $C_\lambda(I)$

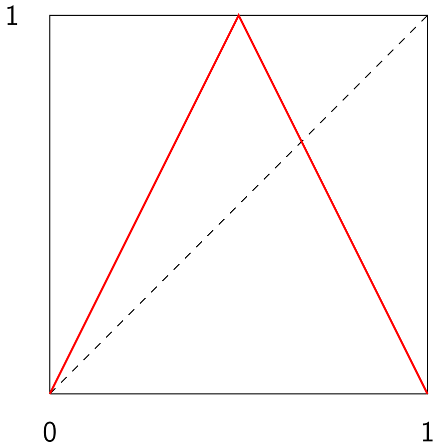
1. λ – the Lebesgue measure on unit circle $\mathbb{S}^1$ and on unit interval I. Let $M \in \{\mathbb{S}^1, I\}$.
2. main spaces

$$C_\lambda(M) := \{f \in C(M);\ \forall\, A \subset M, A \text{ Borel}:\ \lambda(A) = \lambda(f^{-1}(A))\},$$

3. we endow the set $C_\lambda(M)$ with the metric ρ of uniform convergence.
4. Obs: $C_\lambda(M, \rho)$ is a complete metric space.
5. a property P is typical in $(C_\lambda(M, \rho) \equiv$ the set of all maps with the property P is residual ($\equiv$ contains dense G_δ set), maps bearing a typical property are called generic.

A basic example

Ex: $T_2 : M \to M$ defined by $T_2(x) = \min_{x \in M}\{2x, 2-2x\}$.



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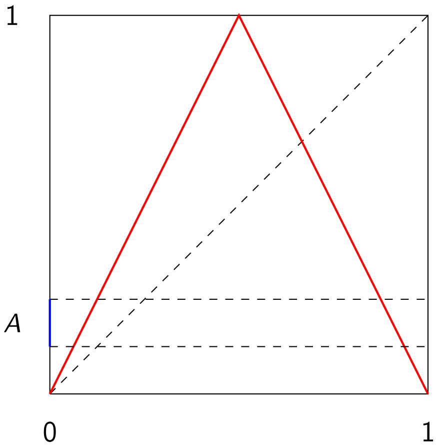

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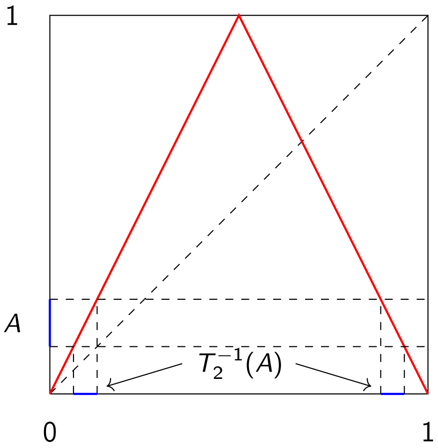

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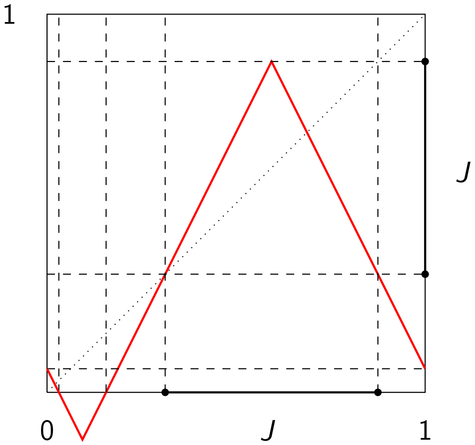

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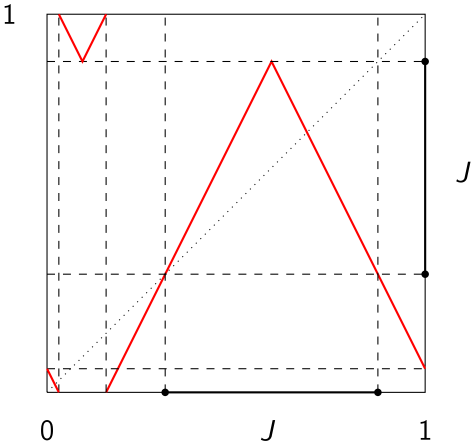

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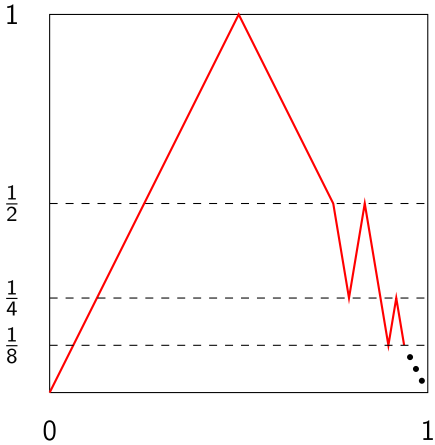

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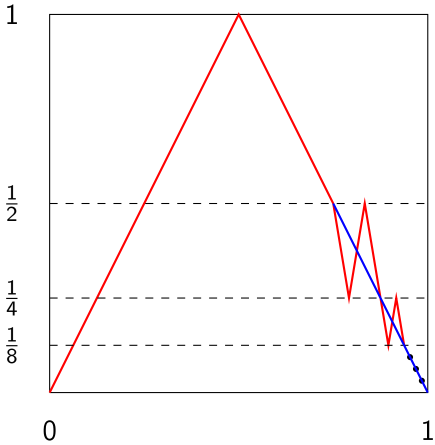


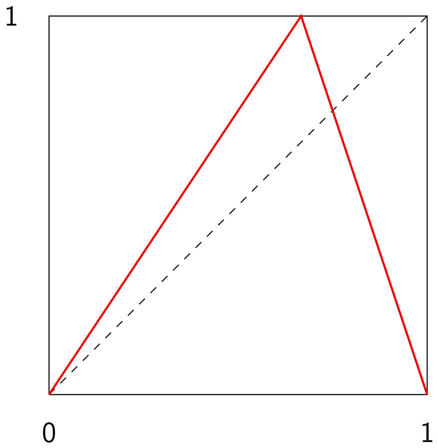
A useful basic criterion

Observation: A map $f \in C(M)$ is in $C_\lambda(M)$ $\iff$ for any nondegenerate arc $J \subset M$,

$$\sum_{K \in \mathrm{Comp}(f^{-1}(J))} \frac{\lambda(K)}{\lambda(J)} = 1, \tag{1}$$

where $\mathrm{Comp}(f^{-1}(J))$ denotes the set of all connected components of $f^{-1}(J)$.

Another example



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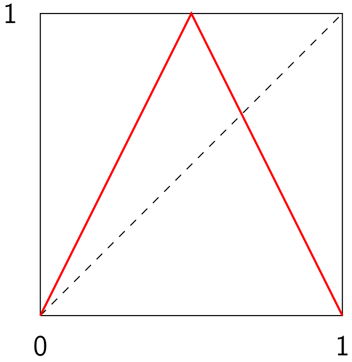

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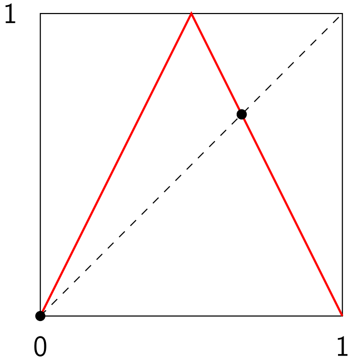

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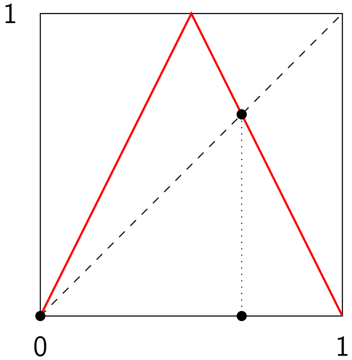

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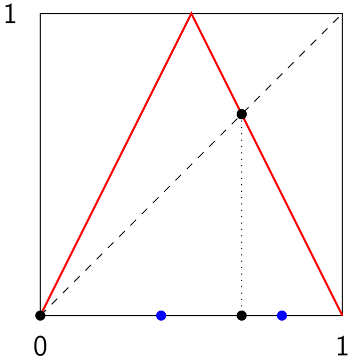

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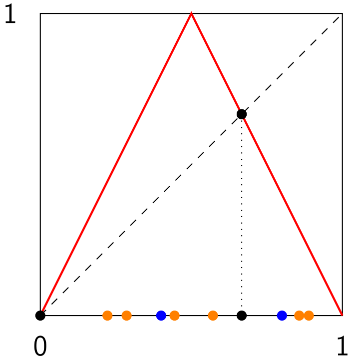

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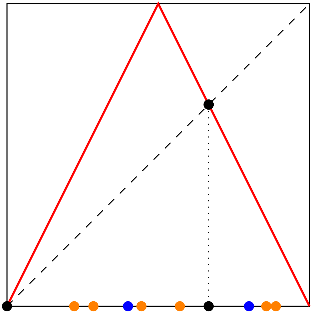

Denote by $\mathrm{Per}(f,n)$ the set of periodic points of period n of $f : M \to M$ and let $\mathrm{Per}(f) = \cup_{n\in\mathbb{N}}\mathrm{Per}(f,n)$. $\overline{\mathrm{Per}(T_2)} = M$.

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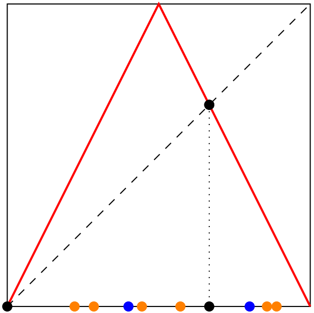

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Denote by $C_{DP}(M)$ the set of maps $f \in C(M)$ such that $\overline{\mathrm{Per}(f)} = M$. Dense periodicity is a prerequisite for "chaoticity".

Relation to dense periodicity on I

Rem: Let $f \in C(I)$. The following conditions are equivalent.

(i) f preserves a nonatomic probability measure μ with $\mathrm{supp}\ \mu = I$.

(ii) There exists a homeomorphism h of I such that $h \circ f \circ h^{-1} \in C_\lambda(I)$.

(iii) $f \in C_{DP}(I)$.

For $C_\mu(I)$, $\mathrm{supp}\ \mu = I$, nonatomic all topological *"generic properties"* can be translated to the complete metric space $(C_\mu(I), \rho)$

Relation to dense periodicity on $\mathbb{S}^1$

Rem: Let $f \in C(\mathbb{S}^1)$, then (i) and (ii) are equivalent, and (iii) implies (i).

(i) f preserves a nonatomic probability measure μ with $\mathrm{supp}\ \mu = \mathbb{S}^1$.

(ii) There exists a homeomorphism h of $\mathbb{S}^1$ such that $h \circ f \circ h^{-1} \in C_\lambda(\mathbb{S}^1)$.

(iii) $f \in C_{DP}(\mathbb{S}^1)$.

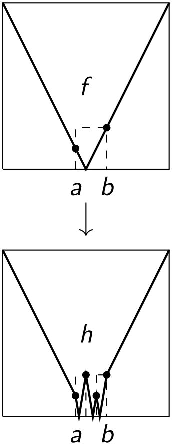
Furthermore, if $\mathrm{Per}(f) \neq \emptyset$, we have (i) implies (iii). Otherwise f is conjugate to an irrational rotation.

Lemma: Let $f \in C(\mathbb{S}^1)$. If $\overline{\mathrm{Rec}(f)} = \mathbb{S}^1$ and $\mathrm{Per}(f) \neq \emptyset$, then $\overline{\mathrm{Per}(f)} = \mathbb{S}^1$.

Periodic points on I

$\mathrm{Fix}(f,k)$ – fixed points of f^k, $\mathrm{Per}(f,k)$ – points of period k for f,

$$\mathrm{Per}(f) = \cup_{k\geq 1}\mathrm{Per}(f,k).$$



Thm (BČOT, 2021): For $C_\lambda(I)$ generic map and $\forall k$:

1. $\mathrm{Fix}(f,k)$ is a Cantor set,
2. $\mathrm{Per}(f,k)$ is a relatively open dense subset of $\mathrm{Fix}(f,k)$,
3. $\mathrm{Fix}(f,k)$ has Hausdorff dimension and lower box dimension zero.
   - In particular, $\mathrm{Per}(f,k)$ has Hausdorff dimension and lower box dimension zero,
   - the Hausdorff dimension of $\mathrm{Per}(f)$ is zero.
4. $\mathrm{Per}(f,k)$ and thus also $\mathrm{Fix}(f,k)$ both have upper box dimension one.

By Schmeling and Winkler (1995) upper box dimension of a graph of a generic map from $C_\lambda(I)$ is 2.

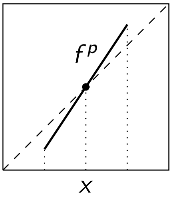
Periodic points on $\mathbb{S}^1$

$C_{\lambda,d}(\mathbb{S}^1)$ – continuous circle maps of degree d preserving Lebesgue measure λ.

1. when $d \in \mathbb{Z} \setminus \{1\}$, generic maps in $C_{\lambda,d}(\mathbb{S}^1)$ satisfy Theorem from the previous slide.
2. the case of $C_{\lambda,1}(\mathbb{S}^1)$ is more delicate, since the set contains irrational rotations.
   2.1 in particular, there are open sets U in $C_{\lambda,1}(\mathbb{S}^1)$ with $\mathrm{Per}(k, f) = \emptyset$ for every $f \in U$.

Circle maps of degree 1

$$C_p := \{f \in C_{\lambda,1}(\mathbb{S}^1) : f \text{ has a transverse } x \in \mathrm{Per}(f,p)\}.$$



Thm (BČOT, 2021): For $\overline{C_p}$ generic map and $\forall k \geq 1$:

1. $\mathrm{Fix}(f, kp)$ is a Cantor set,
2. $\mathrm{Per}(f, kp)$ is a relatively open dense $\subset \mathrm{Fix}(f, kp)$,
3. $\mathrm{Fix}(f, kp)$ has Hausdorff dimension and lower box dimension zero.
   - In particular, $\mathrm{Per}(f, kp)$ has Hausdorff dimension and lower box dimension zero,
   - the Hausdorff dimension of $\mathrm{Per}(f)$ is zero.
4. $\mathrm{Per}(f, kp)$ and thus also $\mathrm{Fix}(f, kp)$ both have upper box dimension one.

The remainder $C_{\lambda,1}(\mathbb{S}^1) \setminus \cup_{p\geq 1}\overline{C_p}$ consists of irrational circle rotations.

Shadowing properties

1. (X, T) – dynamical system
2. a sequence $(x_n)_{n\geq 1}$ is a δ-pseudo orbit if $d(T(x_n), x_{n+1}) < \delta$ for $n \geq 1$.
3. a point $z \in X$ ε-traces δ-pseudo orbit if $d(T^n(z), x_n) < \varepsilon$ for $n \geq 1$.
4. (X, T) has shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ε-traced.
5. (X, T) has limit shadowing property if for every sequence $(x_n)_{n\geq 1} \subset X$ so that

$$d(T(x_n), x_{n+1}) \to 0 \text{ when } n \to \infty$$

(asymptotic pseudo orbit) there exists $z \in X$ such that

$$d(T^n(z), x_n) \to 0 \text{ as } n \to \infty.$$

Shadowing results

S-limit shadowing: for every $\varepsilon > 0$ there exists $\delta > 0$ so that

1. for every δ-pseudo orbit $\mathbf{y} := (y_n)_{n\geq 1}$ we can find a corresponding point $z \in X$ which ε-traces $\mathbf{y}$,
2. for every asymptotic δ-pseudo orbit $\mathbf{y} := (y_n)_{n\geq 1}$ of f, there is $z \in X$ which ε-traces $\mathbf{y}$ and $\lim_{n\to\infty} d(y_n, f^n(z)) = 0$.

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Thm (Bobok, Č., Oprocha, Troubetzkoy, 2022/2023): Shadowing is a typical property in $C_\lambda(I)$. S-limit shadowing is a typical property for maps from $C_\lambda(\mathbb{S}^1)$.

Cor: Limit shadowing and shadowing are typical properties for maps from $C_\lambda(\mathbb{S}^1)$.

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Cor: Limit shadowing and shadowing are typical properties for maps from $C_\lambda(\mathbb{S}^1)$.

Up to our knowledge, it is the first result showing that limit shadowing is typical (previously, only results on density were known - e.g. on $n \geq 2$-dimensional manifolds)

Invitation to the upcoming talk

Thm (Č., Oprocha, 2023) The sets of all maps $\mathcal{T}_{\mathbb{S}^1} \subset C_\lambda(\mathbb{S}^1)$ and $\mathcal{T}_I \subset C_\lambda(I)$ satisfying "crookedness condition" are generic.

Rem: This implies that inverse limit $\varprojlim(\mathbb{S}^1, f)$ for any $f \in \mathcal{T}_{\mathbb{S}^1}$ with degree 1 is the pseudo-circle and inverse limit $\varprojlim(I, g)$ for any $g \in \mathcal{T}_I$ is the pseudo-arc.

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Piotr will say more about that...

Do generic dynamical properties commute with rotations?

For $\alpha \in [0,1)$ we define the map $r_\alpha \colon \mathbb{S}^1 \to \mathbb{S}^1$ as

$$r_\alpha(x) := x \cdot e^{2\pi i \alpha}, \; x \in \mathbb{S}^1$$

and the operator $T_{\alpha,\beta} \colon C_\lambda(\mathbb{S}^1) \to C_\lambda(\mathbb{S}^1)$ by

$$T_{\alpha,\beta}[f] := r_\alpha \circ f \circ r_\beta.$$

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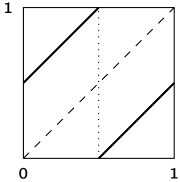
Question: if a property P is typical, does the typical f satisfy property P for "most" (α, β)?

Thm (Athreya, Boshernitzan, 2013): For any interval exchange transformation (IET) f the map $f \circ r_\beta$ is uniquely ergodic for almost every $\beta \in [0,1)$.



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Def: A map $f : X \to X$ is locally eventually onto (leo), if for every open $U \subset X$ there exists $n \geq 1$ such that $f^n(U) = X$.

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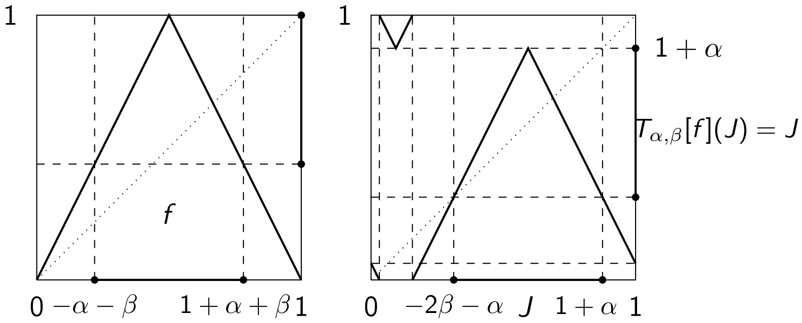
Example: Let f be a full tent map viewed as a circle map. For $\alpha < -\beta \pmod 1$ and $\alpha + \beta > -\frac{1}{2} \pmod 1$ we define the interval $J = J_{\alpha,\beta} := [-2\beta - \alpha, 1 + \alpha] \pmod 1$. We obtain that $T_{\alpha,\beta}[f](J) = J$. Therefore, $T_{\alpha,\beta}[f]$ is not transitive for an open subset of $(\alpha, \beta) \in [0, 1)^2$.



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Thm (Bobok, Č, Oprocha, Troubetzkoy (2022)): There is an open dense set $O \subset C_\lambda(\mathbb{S}^1)$ such that:

1. each $f \in O$ is leo.
2. for each pair $\alpha, \beta \in [0, 1)$, each $f \in O$ the map $T_{\alpha,\beta}[f] \in O$.

Some ideas on the proof of openness of leo

Lemma: Let $\mathcal{A}$ denote the collection of all arcs in $\mathbb{S}^1$. There are positive constants $\kappa_n, \delta_n, \eta_n$ and a dense set of maps $\{h_n\} \subset PA_\lambda(\mathbb{S}^1)$ which satisfy for each $n \geq 1$:

(i) For each $A \in \mathcal{A}$ either $h_n(A) = \mathbb{S}^1$ or

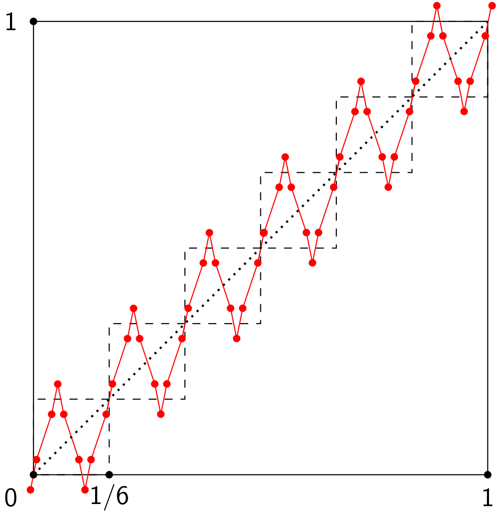
$$\lambda(h_n(A)) \geq (1 + \delta_n)\lambda(A)$$

(ii) each h_n is leo.

(iii) For any arc A with $\lambda(A) > 1 - \eta_n$ it follows that $h_n(A) = \mathbb{S}^1$.

(iv) Let $A \in \mathcal{A}$ and $\lambda(A) < \alpha_n := \min\{\eta_n/2, \kappa_n/3\}$ then $h_n^{-1}(A)$ has at least two non-degenerate components.

An example of a map h_n



Questions

Question 1: Does there exist an open dense set of leo volume preserving noninvertible maps that commute with rotations on higher-dimensional tori?

Question 2: Is s-limit shadowing typical also for $C_\lambda(I)$?

Question 3: Is s-limit shadowing typical also for volume preserving homeomorphisms on manifolds of dimension at least 2?

Question 4: What properties are typical for $C_\lambda(\mathbb{S}^1)$ and $C_\lambda(I)$ if we equip the space with some smoother topology?

Thank you!