Categorical approach to graph limits

Martin Doležal

joint work with Wiesław Kubiś

Institute of Mathematics, Czech Academy of Sciences

45th Conference on Generic Structures
Będlewo, 23–28 October 2023

Category of measurable spaces with Markov kernels as morphisms [Lawvere 1962, Giry 1982]

- objects: measurable spaces
- morphisms: Markov kernels

A Markov kernel from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is a map $\kappa\colon X \times \mathcal{B} \to [0,1]$ s.t.:

- $\kappa(x, \cdot)$ is a probability measure on $(Y, \mathcal{B})$, $x \in X$
- $\kappa(\cdot, B)$ is a measurable map, $B \in \mathcal{B}$

- composition of morphisms: composition of Markov kernels

$$(X, \mathcal{A}) \xrightarrow{\kappa} (Y, \mathcal{B}) \xrightarrow{\kappa'} (Z, \mathcal{C})$$

$$\kappa' \circ \kappa(x, C) = \int_Y \kappa'(y, C)\, d\kappa(x, \cdot), \quad x \in X, C \in \mathcal{C}$$

The identity morphism for $(X, \mathcal{A})$ is

$$1_{(X,\mathcal{A})}(x, A) = \delta_x(A), \quad x \in X, A \in \mathcal{A}$$

□-graphons

Definition

A □-graphon on a probability space $(X, \mathcal{A}, \pi)$ is a finite measure μ on $(X, \mathcal{A})^2$.

In graph terminology:

π = the distribution of vertices

μ = the distribution of edges

Category of □-graphons

- objects: □-graphons
- morphisms:

$$\mu_X \ \ldots \ \square\text{-graphon on } (X, \mathcal{A}, \pi_X)$$
$$\mu_Y \ \ldots \ \square\text{-graphon on } (Y, \mathcal{B}, \pi_Y)$$

A morphism from μ_X to μ_Y is a Markov kernel $\kappa\colon X \times \mathcal{B} \to [0,1]$ from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ such that:

- $\pi_Y(B) = \int_X \kappa(x, B)\, d\pi_X(x), \quad B \in \mathcal{B}$
- $\mu_Y(B_1 \times B_2) = \int_{X^2} \kappa(x_1, B_1)\kappa(x_2, B_2)\, d\mu_X(x_1, x_2), \quad B_1, B_2 \in \mathcal{B}$

- composition of morphisms: composition of Markov kernels

Definition (preorder $\preceq$)

$$\mu_Y \preceq \mu_X \quad \Leftrightarrow \quad \text{there is a morphism from } \mu_X \text{ to } \mu_Y$$

Convergence (essentially defined by Kunszenti-Kovács, Lovász, Szegedy, 2019)

$\mathcal{F}_k = \left(\{1,\dots,k\}, 2^{\{1,\dots,k\}}, \text{the normalized counting measure}\right), \quad k \in \mathbb{N}$

$\square$-graphons on $\mathcal{F}_k \subseteq \mathbb{R}^{k^2}$

Definition (k-shapes)

For a $\square$-graphon μ we define its *k-shape* $\mathbb{S}_k(\mu)$ as

$$\mathbb{S}_k(\mu) = \text{cl}\,\{\nu \preceq \mu : \nu \text{ is a } \square\text{-graphon on } \mathcal{F}_k\}$$

Definition (convergence)

$\mu, \mu_n, n \in \mathbb{N}$ ... $\square$-graphons

$(\mu_n)_{n=1}^{\infty}$ is *convergent* $\Leftrightarrow$ $(\mathbb{S}_k(\mu_n))_{n=1}^{\infty}$ is convergent in the Vietoris topology of compact subsets of $\mathbb{R}^{k^2}$, $k \in \mathbb{N}$

μ is a *limit* of $(\mu_n)_{n=1}^{\infty}$ $\Leftrightarrow$ $\mathbb{S}_k(\mu)$ is the limit of $(\mathbb{S}_k(\mu_n))_{n=1}^{\infty}$ in the Vietoris topology, $k \in \mathbb{N}$

Existence of limit objects

Theorem

Every convergent sequence $(\mu_n)_{n=1}^{\infty}$ of $\square$-graphons has a limit.

Sketch of the proof:

ν ... a $\square$-graphon on a finite probability space $(\Omega, 2^{\Omega}, \pi_{\Omega})$

We say that ν is achievable if, for every $n \in \mathbb{N}$, there is a $\square$-graphon $\gamma_n \preceq \mu_n$ on $(\Omega, 2^{\Omega}, \pi_{\Omega}^{n})$ for some probability measure π_{Ω}^{n} such that

- $\lim_{n\to\infty} \pi_{\Omega}^{n} = \pi_{\Omega}$
- $\lim_{n\to\infty} \gamma_n = \nu$

After passing to a subsequence, there is a countable upward directed set H of achievable $\square$-graphons containing a dense subset of $\lim_{n\to\infty} \mathbb{S}_k(\mu_n)$ for every $k \in \mathbb{N}$.

ν^1, ν^2 achievable $\Rightarrow$ there is $\nu^3 \succeq \nu^1, \nu^2$ such that, after passing to a subsequence, ν^3 is achievable.

Use the diagonal method.

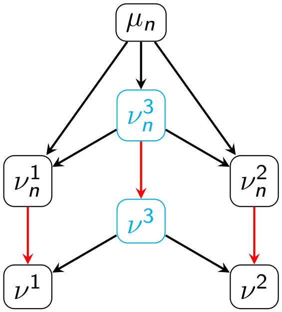

Figure: morphism, convergence

Find a cofinal increasing sequence $(\nu_i)_{i=1}^{\infty}$ of elements of H. Using monotonicity, $(\nu_i)_{i=1}^{\infty}$ has a limit.

ν_i ... a $\square$-graphon on $(\Omega_i, 2^{\Omega_i}, \tilde{\pi}_i)$, $i \in \mathbb{N}$

$Y := \prod_{i=1}^{\infty} \Omega_i$
$\mathcal{B}_Y$ ... the product sigma-algebra on Y

There is a probability measure π_Y on $(Y, \mathcal{B})$ and a $\square$-graphon μ_Y on $(Y, \mathcal{B}, \pi_Y)$ such that

$$\mu_Y = \lim_{i \to \infty} \nu_i.$$

The limit of $(\nu_i)_{i=1}^{\infty}$ is also the limit of $(\mu_n)_{n=1}^{\infty}$.

$\square$