# On Universal Minimal Spaces

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Based on joint works with Eli Glasner, Hanfeng Li, Todor Tsankov and Andy Zucker.

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# Structure theorems for t.d.s

- Let $(G, X)$ be a topological dynamical system (t.d.s). That is:
- $X$ is a compact (Hausdorff) space. $G$ is topological (Hausdorff) group. Action denoted by $gx$ for $g \in G$ and $x \in X$.
- $G$ acts on $X$, $G \curvearrowright X$: $1_G x = x$, $g(hx) = (gh)x$.
- Morphisms: $\phi : (G, X) \to (G, Y)$, where, $\phi : X \to Y$ - equivariant continuous mapping ($\phi(g(x)) = g(\phi(x))$, for every $x \in X$ and $g \in G$):

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \phi\downarrow & & \downarrow\phi \\ Y & \xrightarrow[g]{} & Y \end{array}$$

- *Mostly* $G$ is Polish. The space $X$ is not necessarily metrizable.
- Fundamental question: What is the structure of t.d.s?
- Standing assumption: $(G, X)$ is minimal, that is every **orbit**, $G.x \triangleq \{g.x | g \in G\}$ is dense.
- Known structure theorems: Fustenberg (1963), Ellis-Glasner-Shapiro (1975),Veech (1977),...

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# Universal Minimal Spaces

Let $G$ be a topological group. Key question: Can one capture all minimal $G$ (topological dynamical) systems with one *simple* (read minimal) system?

**Definition**

A universal minimal ($G$)-space (UMS) $M_G$ is a minimal $G$-space with the property that every minimal $G$-space $X$ is a *factor* of $M_G$, i.e., there is a continuous $G$-equivariant map from $M_G$ onto $X$.

**Theorem (Ellis, 1960)**

*Let $G$ be a topological group. There is a unique, up to isomorphism, universal minimal ($G$)-space.*

In addition to Ellis' proof, proofs were given by Auslander (1988) and Uspenskij (2000). Existence is easy: Let $\{M_\alpha\}_{\alpha \in I}$ be an enumeration of all minimal $G$-spaces. Let $M_G$ be a minimal subsystem of $\prod_{\alpha \in I} M_\alpha$.

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- Indeed If $M_1$ and $M_2$ are universal minimal $G$-spaces then by universality we have epimorphisms $\phi_1 : M_1 \to M_2$ and $\phi_2 : M_2 \to M_1$. If in addition $M_1$ is coalescent, then $\phi_2 \circ \phi_1$ must be an isomorphism, and hence $\phi_1$ and $\phi_2$ are isomorphisms.

# A New Short Proof the Uniqueness of the Universal Minimal Space (G. and Li)

- Let $M = X_0$ be universal and minimal. Let $\psi : M \to M$ be a $G$-epimorphism. Fix an ordinal with $|\beta| > |M^2|$.

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- Let $M = X_0$ be universal and minimal. Let $\psi : M \to M$ be a $G$-epimorphism. Fix an ordinal with $|\beta| > |M^2|$.
- Construct a tower of extensions of minimal $G$-spaces $\{X_\alpha\}_{0 \leq \alpha \leq \beta}$, $\phi_{\delta,\alpha} : X_\alpha \to X_\delta$ with $\phi_{\delta,\alpha} = \phi_{\delta,\gamma} \circ \phi_{\gamma,\alpha}$ for $\alpha \geq \gamma \geq \delta$ by means of transfinite induction. If $\alpha$ is a successor ordinal, take an epimorphism $f_\alpha : M \to X_{\alpha-1}$ using the universality of $M$ and define $(G, X_\alpha) = (G, M)$, $\phi_{\alpha-1,\alpha} = f_\alpha \circ \varphi$, and for $\gamma < \alpha - 1$, define $\phi_{\gamma,\alpha} = \phi_{\gamma,\alpha-1} \circ \phi_{\alpha-1,\alpha}$. If $\alpha$ is a limit ordinal, define $X_\alpha$ to be the projective limit of $\{X_\gamma\}_{0 \leq \gamma < \alpha}$, and for $\gamma < \alpha$, define $\phi_{\gamma,\alpha} : X_\alpha \to X_\gamma$ to be the epimorphism coming from the projective limit.

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- **Key property**: For each ordinal $\gamma < \beta$, there exist distinct $x_\gamma, y_\gamma \in X_{\gamma+1}$ with $\phi_{\gamma,\gamma+1}(x_\gamma) = \phi_{\gamma,\gamma+1}(y_\gamma) \triangleq z_\gamma$.

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- For each $\gamma < \beta$, since $\phi_{\gamma+1,\beta}$ is surjective, we can find $\tilde{x}_\gamma, \tilde{y}_\gamma \in X_\beta$ with $\phi_{\gamma+1,\beta}(\tilde{x}_\gamma) = x_\gamma$ and $\phi_{\gamma+1,\beta}(\tilde{y}_\gamma) = y_\gamma$. For any $\gamma < \alpha < \beta$, one has $\phi_{\gamma+1,\beta}(\tilde{x}_\alpha) = \phi_{\gamma+1,\alpha}(z_\alpha) = \phi_{\gamma+1,\beta}(\tilde{y}_\alpha)$, and $\phi_{\gamma+1,\beta}(\tilde{x}_\gamma) = x_\gamma \neq y_\gamma = \phi_{\gamma+1,\beta}(\tilde{y}_\gamma)$, and hence $(\tilde{x}_\alpha, \tilde{y}_\alpha) \neq (\tilde{x}_\gamma, \tilde{y}_\gamma)$.

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- This implies that the map $\{\gamma | \, 0 \leq \gamma < \beta\} \to X_\beta \times X_\beta$ given by $\gamma \mapsto (\tilde{x}_\gamma, \tilde{y}_\gamma)$ is injective, which in turn implies that $|\beta| \leq |X_\beta^2|$.

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- Putting all the inequalities together including our initial choice $|\beta| > |M^2|$, we get $|M^2| < |\beta| \leq |X_\beta^2| \leq |M^2|$, which is impossible.

# Calculation of UMS

If $G$ is compact then $(G, M_G) = (G, G)$.

**Definition**

If $M_G = \{\bullet\}$, then we call $G$ extremely amenable.

Note $G$ is extremely amenable iff every $G$-system has a $G$-fixed point.

**Theorem (Gromov & Milman, 1983)**

*$U(H)$ the unitary group of the infinite-dimensional separable Hilbert space $H$ with the strong operator topology is extremely amenable.*

**Theorem (Pestov, 1998)**

*Let $\mathrm{Homeo}_+(S^1)$ be the group of orientation-preserving homeomorphisms of the circle equipped with the compact-open topology.*

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# Pestov's questions

The revitalization of the universal minimal space theory is due to Vladimr Pestov. He has stated several influential questions in particular in his 2006 AMS University Lecture Series monograph *Dynamics of infinite-dimensional groups: the Ramsey-Dvoretzky-Milman phenomenon*.

**Question (Pestov)**

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- $c \in Exp(Exp(X))$ is a **maximal chain** if for any chain $d$, $c \subset d$ implies $c = d$.

# The Space of Maximal Chains

- **The hyperspace of $X$:** $Exp(X) = \{V \subset X | \emptyset \neq V \text{ is closed}\}$ equipped with the *Vietoris topology*:

$$\langle U_1, U_2, \ldots U_n \rangle = \{F \in Exp(X) | F \subset \bigcup_{k=1}^{n} U_k,\ F \cap U_k \neq \emptyset,\ k = 1 \ldots, n\}$$

- $c \in Exp(Exp(X))$ is a **chain**: $A, B \in c \Rightarrow A \subset B \vee B \subset A$.
- $c \in Exp(Exp(X))$ is a **maximal chain** if for any chain $d$, $c \subset d$ implies $c = d$.
- **The space of maximal chains**:

$$\Phi(X) = \{c \in Exp(Exp(X)) | \, c \text{ is a maximal chain}\}$$

# Results using the maximal chain approach

**Definition**

$(G, X)$ is called 3-transitive if $|X| \geq 3$ and for all distinct $a, b, c \in X$ and distinct $a', b', c' \in X$, there is $g \in G$ so that $ga = a'$, $gb = b'$ and $gc = c'$.

**Theorem (Uspenskij, 2000)**

*Let $G$ be a topological group, then $(G, M_G)$ is not 3-transitive.*

Thus $(\mathrm{Homeo}(Q), M_{\mathrm{Homeo}(Q)}) \neq (\mathrm{Homeo}(Q), Q)$.

**Theorem (Glasner & Weiss, 2003)**

*Let* $\mathrm{Homeo}(K)$ *be the homeomorphism group of the Cantor set equipped with the compact-open topology. Then*

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Using Stone's Duality Theorem a zero-dimensional compact Hausdorff h-homogeneous space $X$ is the Stone dual of a homogeneous Boolean Algebra, i.e. any such space is realized as the space of ultrafilters $B^*$ over a homogeneous Boolean algebra $B$ equipped with the topology given by the base $N_a = \{U \in B^* : a \in U\}$,

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- Every infinite free Boolean algebra is homogeneous. These Boolean algebras correspond by Stone duality to the generalized Cantor spaces, $\{0,1\}^{\kappa}$, for infinite cardinals $\kappa$
- Let $P(\omega)$ be the Boolean algebra of all subsets of $\omega$ (the first infinite cardinal) and let $fin \subset P(\omega)$ be the ideal comprising the finite subsets of $\omega$. Define the equivalence relations $A \sim_{fin} B$, $A, B \in P(\omega)$, if and only if $A \triangle B$ is in $fin$. The quotient Boolean algebra $P(\omega)/fin$ is homogeneous. This Boolean algebra corresponds by Stone duality to the **corona** $\omega^* = \beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone-Čech compactification of $\omega$.

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**Theorem (Glasner & G.)**

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**Corollary:** $U_{Homeo(\omega^*)} = \Phi(\omega^*)$!

# Connected Maximal chains

## Definition

A maximal chain $c \in \Phi(X)$ is connected iff $c$ is connected as a compact subspace of $\mathrm{Exp}(X)$, equivalently, each member of $c$ is connected.

Denote by $\Phi_c(X)$ the space of connected maximal chains.

## Theorem (Gutman, 2008)

*Let $Q$ be the Hilbert cube or a connected manifold of dimension $\geq 2$. Then $(\mathrm{Homeo}(Q), \Phi_c(Q))$ is minimal.*

## Question (Pestov)

*Let $Q$ be the Hilbert cube or a connected manifold of dimension $\geq 2$. Does it hold*

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# UMS of Homeo(high-dimensional manifold) is not metrizable

## Definition

Let $X$ be compact and $U \subset X$ open. Denote by $G_U$ the rigid stabilizer of $U$, i.e., the subgroup of all elements of $G$ that fix all points in $X \setminus U$. A subgroup $G \leq \mathrm{Homeo}(X)$ is called locally transitive if for every open $U \subset X$ and every $x \in U$, $G_U x$ contains a neighborhood of $x$.

## Example

Let X be the Hilbert cube or a closed manifold of dimension 2 or higher. Then any group containing one of the following groups is locally transitive: $\mathrm{Homeo}_0(X)$, $\mathrm{Homeo}_+(X)$ (for $X$ orientable), $\mathrm{Diffeo}_0(X)$ (for $X$ smooth).

## Theorem (G, Tsankov & Zucker, 2021)

*Let $X$ be a closed manifold of dimension at least 2 or the Hilbert cube and let $G$ be a locally transitive subgroup of $\mathrm{Homeo}(X)$. Then the universal minimal flow of $G$ is not metrizable.*

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# Meagre orbits

We concentrate on the case of closed manifold of dimension at least 3 or the Hilbert cube.

**Theorem (Ben Yaacov, Melleray, Tsankov, 2017)**

*Let $G$ be a Polish group. If $M_G$ is metrizable then it has a comeagre orbit.*

**Theorem (Angel, Kechris & Lyons, 2014)**

*Let $G$ be a Polish group. Supposet $(G, X)$ is minimal and $\phi : (G, X) \to (G, Y)$ is a factor map. If the orbit $Gx_0$ is comeagre then the orbit $G\phi(x_0)$ is comeagre.*

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*Let $X$ be a closed manifold of dimension at least 3 or the Hilbert cube and let $G$ be a locally transitive subgroup of* $\mathrm{Homeo}(X)$. *Then $(G, \Phi_c(X)$ has (only) meagre orbits.*

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# Rosendal's Crierion

## Definition

Let $G$ be a Polish group. By the Birkhoff-Kakutani theorem it admits a right-invariant metric $d$. Denote:

$$G_\varepsilon = \{g \in G : d(g, 1_G) < \varepsilon\}.$$

## Theorem (Rosendal)

*Let $G$ be a Polish group, and suppose $Y$ is a Polish, topologically transitive $G$-space. Then the following are equivalent:*

1. *All orbits in $Y$ are meagre;*
2. *There is $\varepsilon > 0$ and a non-empty open $U \subseteq Y$ such that for any non-empty open $V \subseteq U$, there are non-empty open $W_0, W_1 \subseteq V$ such that $(G_\varepsilon W_0) \cap W_1 = \emptyset$.*

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# The Pseudoarc

## Definition

A continuum is a compact, connected space. An open cover $U_0, \ldots, U_{n-1}$ of a continuum is called a chain cover if for all $i, j < n$,

$$U_i \cap U_j \neq \emptyset \iff |i-j| \leq 1.$$

A continuum is chainable if every open cover admits a refinement that is a chain cover. A continuum is indecomposable if it is not the union of two proper subcontinua. It is hereditarily indecomposable if every subcontinuum is indecomposable.

## Theorem (Bing, 1951)

*There is a unique, up to homeomorphism, non-degenerate (having more than one point), chainable, hereditarily indecomposable continuum.*

This space is referred to as the **pseudoarc** $P$.

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# UMS of Homeo(Pseudoarc)

Let $X$ be a compact space. Note that given a maximal chain $c \in \Phi_c(X)$, it has a minimal element, referred to as its root, $r(c)$, which is a singleton.

**Proposition**

*Let $P$ be the pseudoarc. It holds $\Phi_c(P) = P$.*

**Proof.**

Let $c_1, c_2 \in \Phi_c(P)$. Assume $r(c_1) = r(c_2) := r$. We claim $c_1 = c_2$. Indeed let $C_1 \in c_1$ and $C_2 \in c_2$. As $r \in C_1 \cap C_2$ then $C_1 \cup C_2$ is a continuum. By its indecomposability either $C_1 \subset C_2$ or vice versa. □

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