Generics in invariant subsets of automorphisms of homogeneous structures

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A standard question

Let M be countable ultrahomogeneous.
$\mathrm{Aut}(M)$ = Polish group (pointwise convergence topology).

Standard Question: *When does the space* $\mathrm{Aut}(M)$ *contain a conjugacy class which is* **comeagre** *in* $\mathrm{Aut}(M)$ *?*
(contains the intersection of a countable family of dense open subsets)

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Setup

Let $\rho \in \mathrm{Aut}(M)$, $\rho^{\mathrm{Aut}(M)}$ = conjugacy class of ρ.
Let $\mathcal{C}_\rho = cl(\rho^{\mathrm{Aut}(M)})$, where cl = topological closure in $\mathrm{Aut}(M)$.

Question: *When does the space $\mathcal{C}_\rho$ contain a conjugacy class of* $\mathrm{Aut}(M)$ *which is comeagre in $\mathcal{C}_\rho$?*

For $\mathcal{C} \subseteq_{cl} \mathrm{Aut}(M)$, γ is **generic** in $\mathcal{C}$ if $\gamma^{\mathrm{Aut}(M)}$ is comeagre in $\mathcal{C}$.

Description of all closed subsets $\mathcal{C} \subseteq \mathrm{Aut}(M)$ which are invariant under conjugacy in $\mathrm{Aut}(M)$ and have generics.

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JEP and AP

Let $\mathcal{P}$ be the set of all finite partial isomorphisms of M.
The set $\mathcal{P}$ is ordered by the relation of extension of maps: $\subseteq$.
Let $\mathcal{P}' \subset \mathcal{P}$ be invariant under the natural action of $\mathrm{Aut}(M)$ on $\mathcal{P}$.

$\mathcal{P}'$ has the **joint embedding property** if for any $p_1, p_2 \in \mathcal{P}'$ there is $p_3 \in \mathcal{P}'$ and $\alpha \in \mathrm{Aut}(M)$ such that $p_1 \subseteq p_3$ and $\alpha(p_2) \subseteq p_3$.

$\mathcal{P}' \subseteq \mathcal{P}$ has the **amalgamation property** if

$$\forall p_0, p_1, p_2 \in \mathcal{P}' \, (\, p_0 \subseteq p_1 \wedge p_0 \subseteq p_2 \to \exists p_3 \in \mathcal{P}'$$

$$\exists \alpha \in \mathrm{Aut}(M \, / \, \mathrm{Dom}(p_0) \cup \mathrm{Rng}(p_0)) \, (p_1 \subseteq p_3 \wedge \alpha(p_2) \subseteq p_3).$$

CAP

- $\mathcal{P}' \subseteq \mathcal{P}$ has the **cofinal amalgamation property**

if for any $p_0 \in \mathcal{P}'$ there is an extension $p_0' \in \mathcal{P}'$ such that

$$\forall p_1, p_2 \in \mathcal{P}'(p_0' \subseteq p_1 \wedge p_0' \subseteq p_2 \rightarrow \exists p_3 \in \mathcal{P}'$$

$$\exists \alpha \in \mathrm{Aut}(M/\mathrm{Dom}(p_0') \cup \mathrm{Rng}(p_0'))\ (p_1 \subseteq p_3 \wedge \alpha(p_2) \subseteq p_3)).$$

WAP

- $\mathcal{P}' \subseteq \mathcal{P}$ has the **weak amalgamation property**

if for any $p_0 \in \mathcal{P}'$ there is an extension $p'_0 \in \mathcal{P}'$ such that

$$\forall p_1, p_2 \in \mathcal{P}'(p'_0 \subseteq p_1 \wedge p'_0 \subseteq p_2 \rightarrow \exists p_3 \in \mathcal{P}'$$

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Existence of generics. Open Question

$\mathcal{C}_\rho = cl(\rho^{\mathrm{Aut}(M)})$

$\mathcal{P}_\rho = \{p \in \mathcal{P} : p$ extends to an automorphism from $\mathcal{C}_\rho\}$.

Fact. *The set $\mathcal{C}_\rho$ has a generic automorphism if and only if the family $\mathcal{P}_\rho$ has WAP.*

AP $\Rightarrow$ CAP $\Rightarrow$ WAP

Can WAP be replaced by CAP in this formulation?

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Another open question

Is it true that when $\mathrm{Aut}(M)$ *has a generic automorphism then the family* $\mathcal{P}$ *has CAP?*

In this case there is $\rho \in \mathrm{Aut}(M)$ such that $\mathcal{C}_\rho = \mathrm{Aut}(M)$, $\mathcal{P}_\rho = \mathcal{P}$ and $\mathcal{P} \models \mathrm{WAP} \wedge \mathrm{JEP}$ [1]. Arrive at the equivalen question:

Is it true that $\mathcal{P} \models$ *WAP* $\wedge$ *JEP implies* $\mathcal{P} \models$ *CAP ?*

It is known that there are M such that $\mathcal{P}$ has JEP $\wedge \neg$ WAP.

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[1] when $\mathcal{C} \subseteq_{closed} \mathrm{Aut}(M)$ and $\mathcal{P}_{\mathcal{C}} = \{p \in \mathcal{P} : p \text{ extends to some } \alpha \in \mathcal{C}\}$ then JEP is equivalent to the property that $\mathcal{C}$ has a dense conjugacy class.

Another open question

Is it true that when Aut(M) *has a generic automorphism then the family* $\mathcal{P}$ *has CAP?*

In this case there is $\rho \in$ Aut(M) such that $\mathcal{C}_\rho =$ Aut(M), $\mathcal{P}_\rho = \mathcal{P}$ and $\mathcal{P} \models$ WAP $\wedge$ JEP [1]. Arrive at the equivalen question:

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Why ultrahomogeneous?

[Pabion + KKKP] : M.Pouzet's example.
Structure $(\mathbb{Q}, R)$, where $R(x, y, z) \Leftrightarrow x < y \wedge x < z \wedge y \neq z$.

- $<$ and R are interdefinable and $\mathrm{Aut}(\mathbb{Q}, <) = \mathrm{Aut}(\mathbb{Q}, R)$;
- for any ρ the set $\mathcal{C}_\rho$ has a generic automorphism (see below);
- $(\mathbb{Q}, R)$ is weakly homogeneous but not cofinitely homogeneous (i.e. not ultahomogeneous) [KKKP].

Viewing $\mathbb{Q}$ as $\mathbb{Q} \dot{\cup} \mathbb{Q}$ let $\rho \in \mathrm{Aut}(\mathbb{Q}, <)$ be defined by $\rho(x) = x + 1$ on the first copy of $(\mathbb{Q}, <)$ and be the identity on the second copy. Then any $p \in \mathcal{P}_\rho$ having two fixed points does not have an extension to an amalgamation base.

Countable ultrahomogeneous p.o.sets

Countable ultrahomogeneous partially ordered sets:

Let $1 \leq n \leq \omega$ and let $[n] = \{m \in \omega | m \leq n\}$ (is viewed as an antichain).

Let $B_n = [n] \times \mathbb{Q}$ (an ultrahomogeneous p.o.set w.r. to

$$(a, q) < (b, q') \Leftrightarrow a = b \wedge q < q').$$

Let $C_n = B_n$ but the ordering is defined by

$$(a, q) < (b, q') \Leftrightarrow q < q'.$$

Schmerl: *Any countable ultrahomogeneous p.o.set is isomorphic to $[n]$, B_n, C_n, $1 \leq n \leq \omega$, or to the countable universal ultrahomogeneous p.o.set D.*

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Highly homogeneous structures

A group $G \leq Sym(\omega)$ is **highly homogeneous** if for any pair of fnte A and $B \subset \omega$ of the same size thre is $g \in G$ taking A onto B. A countable structure M is called highly homogeneous if $Aut(M)$ is highly homogeneous .

The ordering of the rationals $(\mathbb{Q}, <)$ and the structures $(\mathbb{Q}, B)$, $(\mathbb{Q}, Cr)$ and $(\mathbb{Q}, S)$ are highly homogeneous, where the **linear betweenness relation** associated with $(\mathbb{Q}, <)$:

$$B(x; y, z) \Leftrightarrow (y < x < z) \vee (z < x < y),$$

the **circular order** on the rationals $\mathbb{Q}$ is defined by

$$Cr(x, y, z) \Leftrightarrow (x < y < z) \vee (z < x < y) \vee (y < z < x).$$

The quaternary **separation relation** S on $\mathbb{Q}$ is induced by the permutations of $\mathbb{Q}$ which preserve or reverse Cr.

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Classification

Cameron: These examples together with S_∞ are the only countable highly homogeneous structures.

They are ultrahomogeneous with respect to their natural languages.

A conjecture

Let M be a countable highly homogeneous structure or an ultrahomogeneous partially ordered set and
let $\rho \in \mathrm{Aut}(M)$ and $\mathcal{C}_\rho = cl(\rho^{\mathrm{Aut}(M)})$,
$\mathcal{P}_\rho = \{p \in \mathcal{P} : p$ extends to an automorphism from $\mathcal{C}_\rho\}$.

Early conjecture:
$\mathcal{C}_\rho$ has a generic element.
$\mathcal{P}_\rho$ satisfies CAP. *In particular CAP $\Leftrightarrow$ WAP.*

Comment: Still a conjecture for highly homogeneous structures.
The equivalence CAP $\Leftrightarrow$ WAP is still a conjecture.

Results

The conjecture holds in the following cases.

I. The case of highly homogeneous structures:

$M = \omega$, $M = (\mathbb{Q}, <)$, $M = (\mathbb{Q}, B)$, $M = (\mathbb{Q}, S)$ or $M = (\mathbb{Q}, Cr)$ and $\rho \in \mathrm{Aut}(M)$ does not have periodic points.

II. The case of countable ultrahomogeneous partially ordered sets.

$M = [\omega]$, $M = B_n$ with $n \in \omega \cup \{\omega\}$ or $M = C_n$ with $n \in \omega$.

Remaining cases

The cases
$M = D$, or
$M = (\mathbb{Q}, Cr)$ and $\rho \in \mathrm{Aut}(M)$ has periodic points,
are completely open.

In the case $M = C_\omega$ *there is* $\rho \in \mathrm{Aut}(M)$ *without generics in* $\mathcal{C}_\rho$.

The equivalence CAP $\Leftrightarrow$ WAP is still a conjecture.

KKKP-Pabion-Pouzet's example

Structure $(\mathbb{Q}, R)$, where $R(x, y, z) \Leftrightarrow x < y \wedge x < z \wedge y \neq z$.
Corollary. If there is no cofinal segment of $\mathbb{Q}$ which is fixed by ρ pointwise, then $\mathcal{P}_\rho$ has CAP.

Orbitals in $(\mathbb{Q}, <)$

Let $\gamma \in \mathrm{Aut}(\mathbb{Q}, <)$ and $a \in \mathbb{Q}$. The **orbital** of γ containing a:

$$\{q \in \mathbb{Q} \,|\, (\exists m, n \in \mathbb{Z})(\gamma^n(a) \leq q \leq \gamma^m(a))\}$$

(a singleton or an open interval).

The **parity function** $\wp_\gamma : \mathbb{Q} \to \{+, -, 0, \}$ of γ is defined as follows:

$$\wp_\gamma(x) = \begin{cases} - & \text{if} \quad \gamma(x) < x \\ 0 & \text{if} \quad \gamma(x) = x \\ + & \text{if} \quad \gamma(x) > x. \end{cases}$$

The parity of an orbital O is $\wp_\gamma(a)$ with $a \in O$.

Conjugacy classes of the group of order-preserving permutations of $\mathbb{Q}$

Let $(\mathcal{O}_\gamma, \preceq_\gamma)$ be the ordering of γ-orbitals.
It is colored by $\wp$.

A classical result of Schreier and Ulam (according to Holland) says that

> γ_1, γ_2 *are conjugate in* $\mathrm{Aut}(\mathbb{Q}, <)$ *if and only if* $(\mathcal{O}_{\gamma_1}, \preceq_{\gamma_1})$ *and* $(\mathcal{O}_{\gamma_2}, \preceq_{\gamma_2})$ *are isomorphic by an isomorphism preserving parity of the orbitals.*

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Colored orders

Let $\Delta = \{+, -, 0\}$.
Δ-Colored ordering = a linear ordering $(L, \leq)$ together with a function $\kappa : L \to \Delta$.

Given $(L, <, \kappa)$ and $(L', <, \kappa)$ a map $\phi : L \to L'$ is called a **κ-homomorphism** if for any $x_1, x_2, x \in L$

$$\kappa(x) = \kappa'(\phi(x)),$$

$$x_1 \leq x_2 \Rightarrow \phi(x_1) \leq' \phi(x_2)$$

and ϕ is injective on the set of all elements of color 0.

The structure of the proof for Aut($\mathbb{Q}, <$)

Fix $\rho \in$ Aut($\mathbb{Q}, <$) and consider *Age*$((\mathcal{O}_\rho, \prec, \wp_\rho))$ with respect to $\wp$*-homomorphisms*.

Theorem 1. *The category Age*$((\mathcal{O}_\rho, \prec, \wp_\rho))$ *has CAP.*

Theorem 2. $\mathcal{P}_\rho$ *has CAP. (i.e.* $\mathcal{C}_\rho$ *has a generic element.)*

Example. When ρ has only $+$-orbitals, then $x \to x + 1$ is generic in $\mathcal{C}_\rho$.

Key notions: The **canonical decomposition** of $(\mathcal{O}_\rho, \prec, \wp_\rho)$ and **its isomorphism type**.

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We need new colors

Let $(L, \leq, \kappa)$ be a colored linear ordering where $\kappa : L \to \{+, -, 0\}$. Let L_0 be a convex subset of L and $\varepsilon \in \{+, -, 0\}$.

- We say that L_0 is of **type** $\infty_{+,-,0}$ if for each natural number n there is a sequence $a_1 < a_2 < \ldots < a_{3(n+1)}$ in L_0 such that

$$\kappa(a_{3i+1}) = + \,,\ \kappa(a_{3i+2}) = - \,,\ \kappa(a_{3i+3}) = 0 \ \text{ where } i \leq n.$$

When $\gamma \in \mathrm{Aut}(\mathbb{Q})$ *and* $(\mathcal{O}_\gamma, \prec)$ *is of type* $\infty_{+,-,0}$*, then*

$$\mathcal{C}_\gamma = \mathrm{Aut}(\mathbb{Q}, <).$$

3 colors $\Rightarrow$ 7 colors

- We say that L_0 is of **type** ε if $\kappa(L_0) = \{\varepsilon\}$ and, moreover, in the case $\varepsilon = 0$ the set L_0 is a singleton.
- We say that L_0 is of **type** ∞_0 if L_0 is infinite and $\kappa(L_0) = \{0\}$.
- Let $\varepsilon' \in \{+, -, 0\} \setminus \{\varepsilon\}$. We say that L_0 is of **type** $\infty_{\varepsilon\varepsilon'}$ if $\kappa(L_0) = \{\varepsilon, \varepsilon'\}$ and for each natural number n there is a sequence $a_1 < a_2 < \ldots < a_{2(n+1)}$ in L_0 such that

$$\kappa(a_{2i+1}) = \varepsilon\,,\ \kappa(a_{2i+2}) = \varepsilon', \text{ where } i \leq n.$$

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7 colors

We now introduce new colors and some partial ordering of them. Let

$$\chi = \{\{+\}, \{-\}, \{0\}, \infty_0\} \cup \{\infty_{\varepsilon_1\varepsilon_2} \,|\, \varepsilon_1\varepsilon_2 \in \{+-, +0, -0\}\}.$$

The ordering of χ corresponds to the relation $\subseteq$.
(In particular when $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$ we put $\{\varepsilon\} \subset \infty_{\varepsilon_1,\varepsilon_2}$.)
We also put $\{0\} \subset \infty_0 \subset \infty_{+0}$ and $\infty_0 \subset \infty_{-0}$.

Note that

$$|\chi| = 7.$$

Orderings colored in 7 colors

Let $(L, \leq, \kappa)$ be a $\{+, -, 0\}$-colored linear ordering. To each decomposition $L = \dot{\bigcup}\{L_i \mid i \leq n\}$ into finitely many intervals of types

$$\tau \in \{+, -, 0, \infty_0\} \cup \{\infty_{\varepsilon_1\varepsilon_2} \mid \varepsilon_1\varepsilon_2 \in \{+-, +0, -0\}\}$$

we associate a χ-coloring $\hat{\kappa}$ of the family $\{L_i \mid i \leq n\}$ as follows:

$\hat{\kappa}(L_i) = \{\tau\}$ *if* L_i *is of type* $\tau \in \{+, -, 0\}$
$\hat{\kappa}(L_i) = \tau$ *if* L_i *is of type* $\tau \in \{\infty_0\} \cup \{\infty_{\varepsilon_1\varepsilon_2} \mid \varepsilon_1\varepsilon_2 \in \{+-, +0, -0\}\}$.

As a result the family $\mathcal{L} = \{L_i \mid i \leq n\}$ becomes a finite χ-colored ordered set where the ordering is just $\prec$.

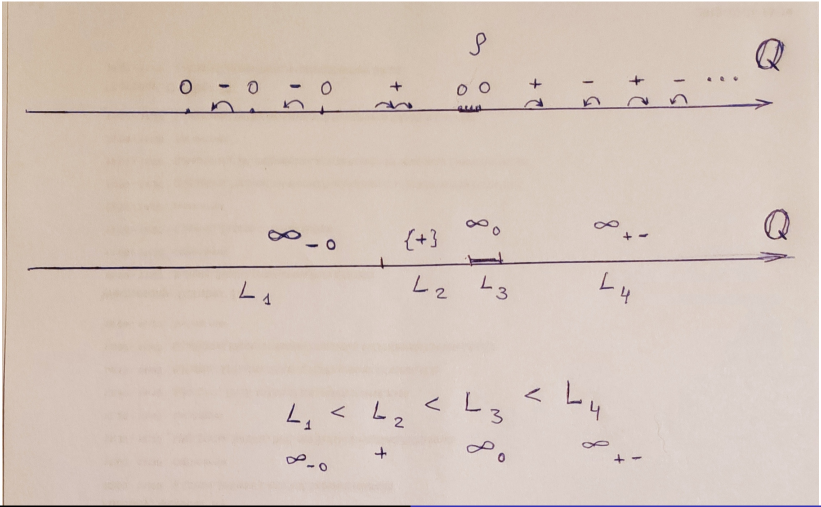
Canonical decomposition

Let $(L, \leq, \kappa)$ be a $\{+, -, 0\}$-colored linear ordering.
Assume that $\mathcal{L}$ is not of type $\infty_{+,-,0}$.

Then there is a decomposition $L = L_1 \dot\cup L_2 \dot\cup \ldots \dot\cup L_n$ into finitely many intervals such that the corresponding χ-colored ordering $(\mathcal{L}, \preceq, \hat{\kappa})$ is *canonical*.

Furthermore, for any two finite decompositions of L defining *canonical* χ-colored orderings, these orderings are isomorphic.

Example of canonical decomposition



Automorphisms of the circular ordering

Let $\alpha \in \mathrm{Aut}(\mathbb{Q}, Cr)$.
View α as an orientation preserving homeomorphism of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.
Let $A : \mathbb{R} \to \mathbb{R}$ be the continuous lift of α (i.e.

$$\alpha(x + \mathbb{Z}) = A(x) + \mathbb{Z}).$$

Principal new case: α does not have periodic points.
Poincaré: In this case there is a rotation $x + r$ with irrational $r > 0$ and a continuous monotone $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ \alpha = h + r$.
It is determined by the map $A^n(x_0) + m \to n \cdot r + m$ for some $x_0, r \in \mathbb{R}$ (this map induces a surjective map

$$\overline{\{A^n(x_0) + m : m, n \in \mathbb{Z}\}} \to \mathbb{R}).$$

If α is topologically transitive, then h is a homeomorphism conjugating α with $x + r$.

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$$\alpha(x + \mathbb{Z}) = A(x) + \mathbb{Z}).$$

Principal new case: α does not have periodic points.
Poincaré: In this case there is a rotation $x + r$ with irrational $r > 0$ and a continuous monotone $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ \alpha = h + r$.
It is determined by the map $A^n(x_0) + m \to n \cdot r + m$ for some $x_0, r \in \mathbb{R}$ (this map induces a surjective map

$$\overline{\{A^n(x_0) + m : m, n \in \mathbb{Z}\}} \to \mathbb{R}).$$

If α is topologically transitive, then h is a homeomorphism conjugating α with $x + r$.

Cofinality

Let $\mathcal{P}$ be the set of all finite partial isomorphisms of $(\mathbb{Q}, Cr)$.
$\mathcal{P}_\alpha = \{p \in \mathcal{P} : p$ extends to an automorphism from $\mathcal{C}_\alpha\}$.

For any $p \in \mathcal{P}_\alpha$ there is an extension $\hat{p} \in \mathcal{P}_\alpha$ which consists of a single orbit.

Lemma. For any $p \in \mathcal{P}_\alpha$ there is a finite partial $p' : \mathbb{S}^1 \to \mathbb{S}^1$ which is a conjugate of p by a $+$-homeomorphism of $\mathbb{S}^1$ and which extends to a rotation $x + r$, $0 \leq r \leq 1$.

Let p'' be a single-orbit restriction of the rotation $x + r$ which extends p'.
It corresponds to a required $\hat{p}$ (by a homeomorphism).

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CAP for *Cr*

Let $\mathcal{P}_{gd} = \{p \in \mathcal{P} : p$ consists of a single orbit $\}$.

Lemma. *$\mathcal{P}_{gd} \cap \mathcal{P}_{\alpha}$ has AP.*

The separation relation

When we view $\rho \in \mathrm{Aut}(\mathbb{Q}, S) \setminus \mathrm{Aut}(\mathbb{Q}, Cr)$ as a homeomorphism of $\mathbb{S}^1$ there are two fixed points, say r_1 and r_2.

As a result $(\mathbb{Q}, S, \rho)$ can be viewed as the partial order B_2 with lines

$$\{x \in \mathbb{Q} : Cr(r_1, x, r_2)\} \text{ or/and } \{x \in \mathbb{Q} \ : \ Cr(r_2, x, r_1)\}.$$

and its automorphism ρ.

Other cases

I. The case of the highly homogeneous structure $M = (\mathbb{Q}, B)$ and the case of the ordering $M = B_n$ with $n \in \omega \cup \{\omega\}$ can be derived using the reslts concrning $M = (\mathbb{Q}, <)$.

II. The case of countable ultrahomogeneous partially ordered sets $M = C_n$ with $n \in \omega$.

Any $\rho \in \mathrm{Aut}(C_n) \Rightarrow$ the natural $\mathbb{Q}$-projection, say $\gamma \in \mathrm{Aut}(\mathbb{Q}, <)$, $(\mathcal{O}_\gamma, \prec)$ has a coloring, say $\wp_\rho$ of

$\Pi = \{+, -\} \cup \{0_f \mid f$ is a cycle function for a permutation $\in S_n\}$.

Remaining cases: $M = C_\omega$ (there are $\mathcal{C}_\rho$ without generics), $M = D$, $M = (\mathbb{Q}, Cr)$ and the rotation number of ρ is rational.