Random graph C^*-algebras

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$$(\forall\, A, B \in \mathcal{E}) \quad A \cong B \Leftrightarrow \mathrm{Ell}(A) \cong \mathrm{Ell}(B)$$

$\mathcal{E}$ contains all C^*-algebras of the form $\lim\limits_{n\in\mathbb{N}} (A_n, \varphi_n)$, where $\forall n \in \mathbb{N}$,

- $A_n \subseteq C(X_n, M_{k_n})$ is a unital subalgebra
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Analogous results in $\mathcal{E}$ can be proved using classification by taking random walks on graphs.

Graph algebras

Graph algebras

A **directed graph** E consists of a vertex set E^0, an edge set E^1 and range and source maps $r, s : E^1 \to E^0$.
A vertex $v \in E^0$ is a **sink** if $s^{-1}(v) = \emptyset$.

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The graph algebra $C^*(E)$ is the universal (unital) C^*-algebra with generators $\{p_v \mid v \in E^0\} \cup \{s_e \mid e \in E^1\}$, where the p_v are mutually orthogonal projections $p^2 = p = p^*$
and the s_e are partial isometries $ss^*s = s$
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satisfying the relations:

- $s_e^* s_f = 0 \quad \forall\, e \neq f \in E^1$
- $s_e^* s_e = p_{r(e)} \quad \forall\, e \in E^1$
- $s_e s_e^* \leq p_{s(e)} \quad \forall\, e \in E^1$
- $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^* \quad \forall\, v \in E^0$ that is not a sink.

Examples $\quad E \quad C^*(E)$

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•	$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \cdots v_{n-1} \xrightarrow{e_{n-1}} v_n$	M_n

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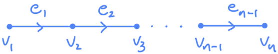
- 

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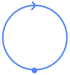
- 
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-  $\qquad$ Cuntz algebra $\mathcal{O}_n$

$$\mathcal{O}_n \cong C^*\left(s_1, \dots, s_n \;\middle|\; s_i^* s_i = 1 = \sum_{i=1}^{n} s_i s_i^*\right)$$

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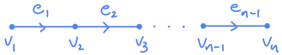	M_n
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Proposition

If E is strongly connected and does not consist of a single cycle, then $C^*(E)$ is $\underbrace{\text{purely infinite}}_{(\forall a, b \neq 0)(\exists x, y)\ xay = b}$ and simple.

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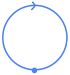
- 
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Proposition

If E is strongly connected and does not consist of a single cycle, then $C^*(E)$ is $\underbrace{\text{purely infinite}}_{(\forall a,b\neq 0)(\exists x,y)\ xay=b}$ and simple.

$\Rightarrow$ $C^*(E) \in \mathcal{E}$ and its stable isomorphism class $[C^*(E)\otimes \mathbb{K}(H)]$ is completely determined by its K-theory.

Proposition

If E is a finite digraph without sinks and A_E is its adjacency matrix $A_E(v,w) = \#\{e \in E^1 \mid s(e)=v,\ r(e)=w\}$, then

$$K_0(C^*(E)) \cong \operatorname{coker}(A_E^t - I)$$

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e.g. $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$, $K_1(\mathcal{O}_n) = 0$

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Q What are typical properties of $\mathbb{D}_{n,q}$ and $C^*(\mathbb{D}_{n,q})$?

Theorem (Erdős - Rényi)

If $q = (\log n + \omega)/n$ for some function $\omega = \omega(n)$, then

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{D}_{n,q} \text{ is strongly connected}) = \begin{cases} 0 & \text{if } \omega \to -\infty \\ e^{-2e^{-c}} & \omega \to c \\ 1 & \omega \to +\infty. \end{cases}$$

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Corollary If q is kept constant as n varies, then $C^*(\mathbb{D}_{n,q})$ is asymptotically almost surely (= with high probability) classifiable by K_0, i.e.

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Reminder $K_0(C^*(\mathbb{D}_{n,q})) = \operatorname{coker} \underbrace{(A(n)^t - I)}_{M(n)}$

$K_1(C^*(\mathbb{D}_{n,q})) = \ker M(n)$ in the absence of sinks

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Corollary $C^*(\mathbb{D}_{n,q})$ is whp a purely infinite simple C^*-algebra with trivial K_1.

Definition (Cuntz polygons)

For $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, let $E_{\bar{m}}$ be the graph with vertex set $\{1, \ldots, n\}$ and edge set $\{l_1, \ldots, l_n\} \cup \{e_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le m_i}}$

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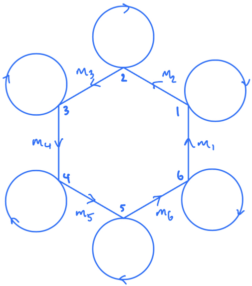

$$A_{\bar{m}} = I + \begin{pmatrix} & m_2 & & & & \\ & & m_3 & & & \\ & & & m_4 & & \\ & & & & m_5 & \\ & & & & & m_6 \\ m_1 & & & & & \end{pmatrix}$$

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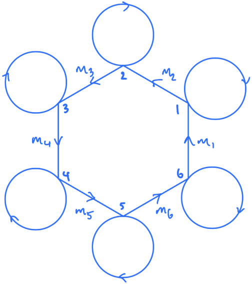

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e.g. $G = \underbrace{\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^4\mathbb{Z}}_{G_3} \oplus \underbrace{\mathbb{Z}/7\mathbb{Z}}_{G_7}$ noncyclic

$G_p = 0$ for all other p

$H = \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/5^2\mathbb{Z} \cong \mathbb{Z}/400\mathbb{Z}$ cyclic

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Heuristically

$$\lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{D}_{n,q})) \text{ cyclic}\right) = \lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{D}_{n,q}))_p \text{ cyclic } \forall p\right)$$

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$$\lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{D}_{n,q})) \text{ cyclic}\right) = \lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{D}_{n,q}))_p \text{ cyclic } \forall p\right)$$

$$\text{"="} \prod_{p \text{ prime}} \underbrace{\lim_{n\to\infty} \mathbb{P}\left(K_0(C^*(\mathbb{D}_{n,q}))_p \text{ cyclic}\right)}_{c_p}$$

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Here, ζ is the Riemann zeta function. Recall Euler's product formula $\zeta(k) = \prod_{p \text{ prime}}(1-p^{-k})^{-1}$.

Empirical data

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The primary tool for computing cokernels of integer matrices is the Smith normal form :

$$M \in M_n(\mathbb{Z}) \rightsquigarrow UMV = D = \mathrm{diag}(d_1, \dots, d_n)$$

where $U, V \in GL_n(\mathbb{Z})$, $d_i \mid d_{i+1}$ $\forall i$

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We recorded tallies of:

- strong connectivity
- nontriviality of K_1
- cyclicity of K_0 and $(K_0)_p$ for several primes p.

The typical sample size was $m = 10^5$.

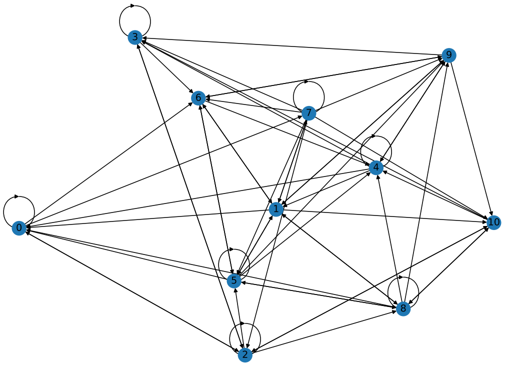

Figure: Sample generated graph $\mathbb{D}_{11,1/2}$

Random regular graph (n=20, r=3)

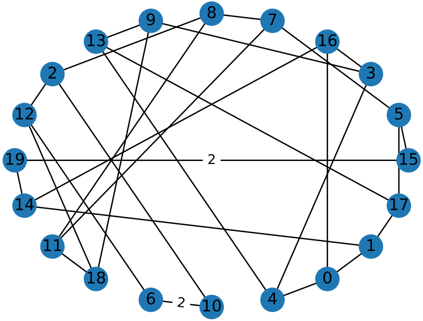

Figure: Sample generated graph $\mathbb{G}_{20,3}$

(n,q)	$\mathbb{D}_{n,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p=2$	$p=3$	$p=5$	$p=7$
$(50,1/2)$	100000	0	84881	86769	98104	99788	99954
$(100,1/2)$	100000	0	85098	86928	98086	99819	99961
$(50,1/3)$	100000	0	84597	86598	97975	99784	99950
$(100,1/3)$	100000	0	84727	86676	98003	99801	99952
$(50,1/4)$	99994	0	84756	86679	98057	99793	99955
$(100,1/4)$	100000	0	84586	86570	97982	99805	99958
$n \to \infty$	$10^5 - O(1/n)$	$10^5(1/\sqrt{2}+o(1))^n$	84694	86636	98022	99794	99951

Table: $\mathrm{C}^*(\mathbb{D}_{n,1/k})$, $n = 50, 100$, $k = 2, 3, 4$

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$q\ (n=100)$	$\mathbb{D}_{100,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p=2$	$p=3$	$p=5$	$p=7$
$3\log n/n$	99993	3	84617	86586	97990	99781	99958
$2\log n/n$	98606	114	84880	86786	98062	99805	99952
$\log n/n$	15623	8829	85713	87481	98183	99801	99941
$\log n/2n$	0	51702	87968	89172	98776	99866	99978

Table: $\mathrm{C}^*(\mathbb{D}_{n,k\log n/n}),\ n=100,\ k=3,2,1,0.5$

(n,q)	$\mathbb{E}_{n,q}$ connected	$K_1 \neq 0$	$(\mathrm{Tor}(K_0))_p$ cyclic				
			all p	$p=2$	$p=3$	$p=5$	$p=7$
$(50, 1/2)$	100000	0	79251	83796	95908	99129	99689
$(100, 1/2)$	100000	0	79463	83990	95806	99182	99713
$(50, 1/3)$	100000	0	79355	83960	95792	99137	99698
$(100, 1/3)$	100000	0	79488	84056	95890	99172	99675
$(50, 1/4)$	99999	0	79349	83804	95919	99215	99688
$(100, 1/4)$	100000	0	79279	83774	95890	99141	99690
$n \to \infty$	$10^5 - O(1/n)$	$o(1)$	79352	83884	95851	99167	99702

Table: $\mathrm{C}^*(\mathbb{E}_{n,1/k})$, $n = 50, 100$, $k = 2, 3, 4$

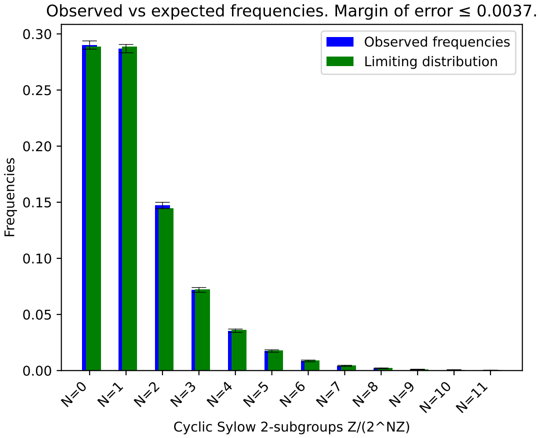

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{D}_{100,1/4}))_2$

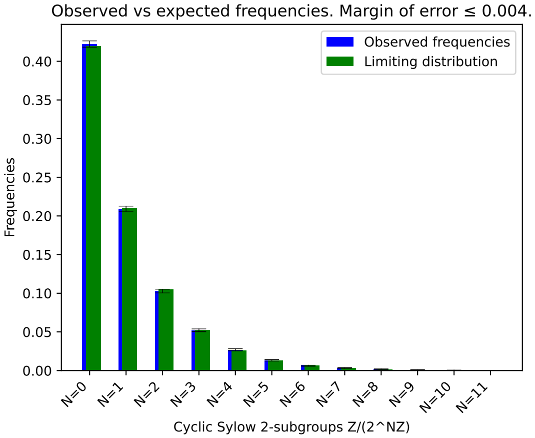

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{E}_{100,1/2}))_2$

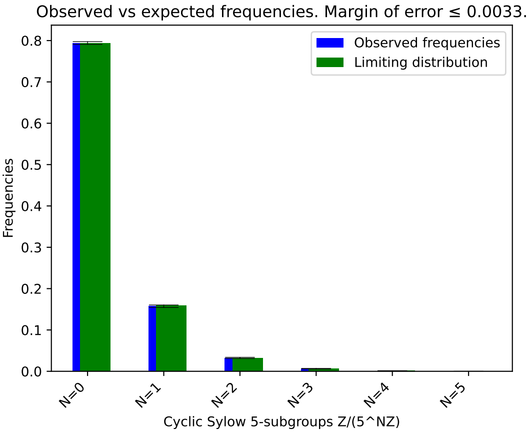

Figure: Frequency distribution for $K_0(\mathrm{C}^*(\mathbb{G}_{200,3}))_5$

Děkuji!