Generic Besicovitch sets in the plane

Tamás Kátay (ELTE, Hungary)

Conference on Generic Structures, Będlewo, October 2023

Motivation

Kakeya's Needle Problem

Question

What is the **minimum area** of a planar set in which a **unit line segment can be turned** by 180° so that it returns to its original position?

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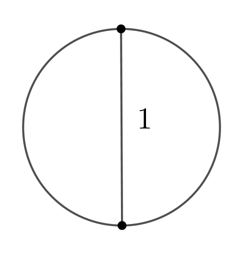

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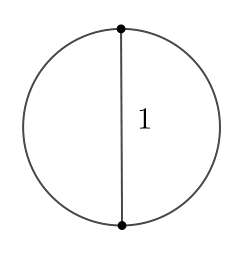

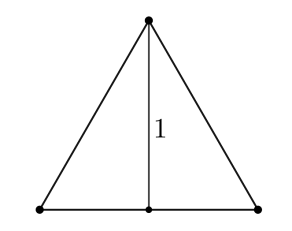

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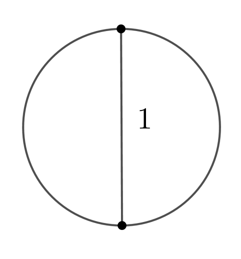

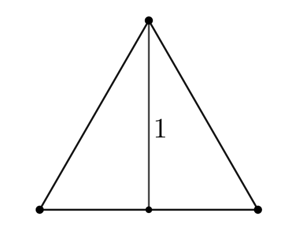

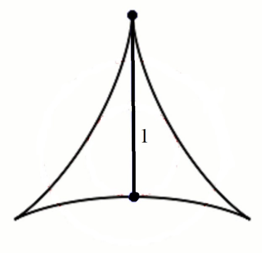

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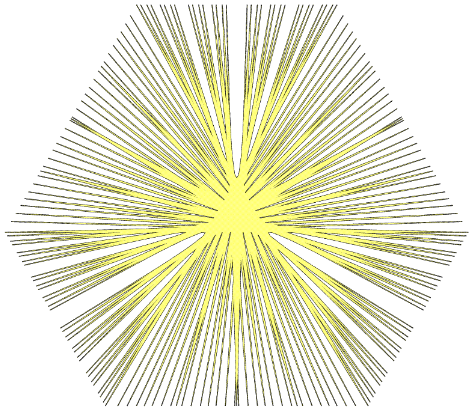

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A subset $B \subseteq \mathbb{R}^n$ is a **Besicovitch set** if it contains a unit line segment in every direction.

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With this definition:

- $\exists$ closed Besicovitch nullset
- the Kakeya Conjecture is open
- the two versions are equivalent (T. Keleti and A. Máthé)

The problem

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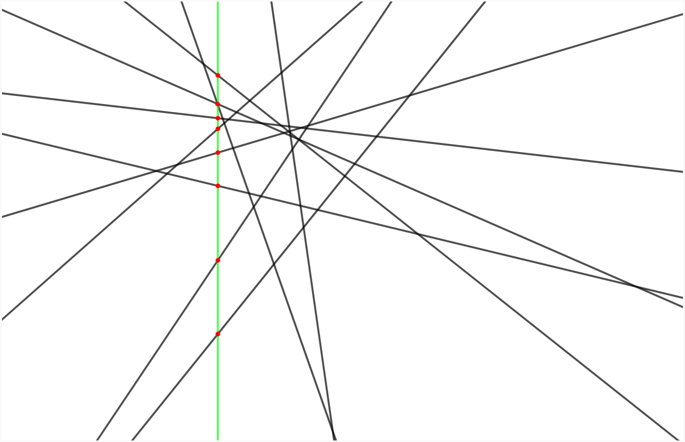

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There exists a Besicovitch set $B = \bigcup \mathcal{L}$ (where $\mathcal{L}$ is a family of lines) in the plane such that:

(1) B is closed;

(2) $\lambda^2(B) = 0$;

(3) for every line $e \notin \mathcal{L}$ we have $\lambda^1(B \cap e) = 0$;

(4) for every line $e \in \mathcal{L}$ we have $\lambda^1(e \cap \bigcup(\mathcal{L} \setminus \{e\})) = 0$.

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Moreover, these properties are generic.

Suffices to have lines with every slope in $[0, 1]$.

The duality method

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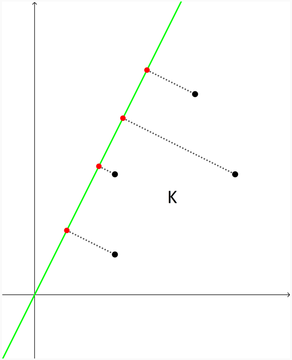
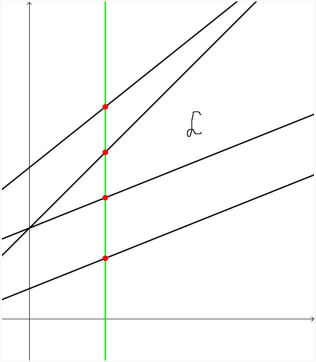
The family $\mathcal{L}$ contains a line with every slope in $[0, 1]$ if and only if $[0, 1] \subseteq \mathrm{proj}_x(K)$.

Duality

Observation 2

The vertical sections of $\bigcup \mathcal{L}$ are scaled copies of (non-vertical) orthogonal projections of K.





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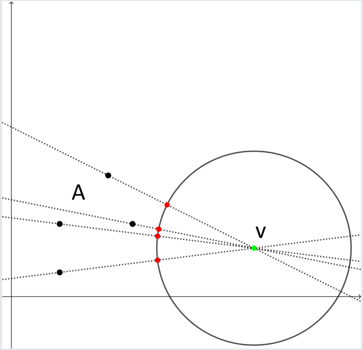
The **radial projection** of a set $A \subseteq \mathbb{R}^2$ from a point v is:



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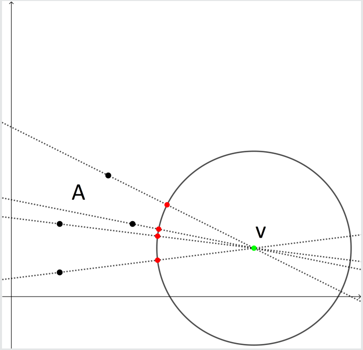


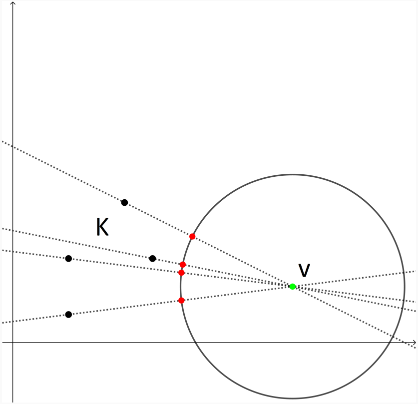
Definition

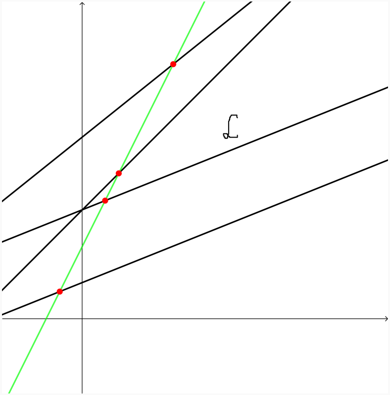
The set $A \subseteq \mathbb{R}^2$ is **invisible** from the point $v \in \mathbb{R}^2$ if the radial projection of A from v is a nullset.

Duality

Observation 3

Let $K \subseteq \mathbb{R}^2$ and let $\mathcal{L}$ be its dual. Then the non-vertical sections of $\bigcup \mathcal{L}$ are locally Lipschitz images of radial projections of K.





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Thus it suffices to prove the following:

Theorem

There exists a compact set $K \subseteq [0,1]^2$ such that:

(1) $\text{proj}_x(K) = [0,1]$;

(2) its orthogonal projection is a nullset in every non-vertical direction;

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The generic code set

The space of code sets

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It suffices to prove *separately* that (2) and (3) are generic properties in $\mathcal{C}$.

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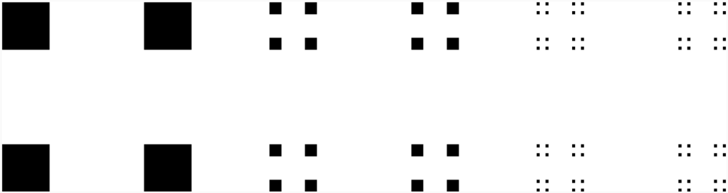

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It does not happen for a **generic** $K \in \mathcal{C}$.

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Is $\bigcup \widetilde{\mathcal{L}}$ Borel?

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The set $\bigcup \widetilde{\mathcal{L}}$ is the image of $\widetilde{K} \times \mathbb{R}$ under the continuous map $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(a, b, t) = (t, at + b)$.

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Follows from the Lusin-Novikov theorem:

Fact

The image of a Borel set under a countable-to-one continuous function is Borel.

Open problem: uniform Besicovitch sets

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The set $\bigcup \widetilde{\mathcal{L}}$ is the image of $\widetilde{K} \times \mathbb{R}$ under the continuous map $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(a, b, t) = (t, at + b)$.

$\implies \bigcup \widetilde{\mathcal{L}}$ is analytic.

Follows from the Lusin-Novikov theorem:

Fact

The image of a Borel set under a countable-to-one continuous function is Borel.

Thus it suffices to prove that $f|_{\widetilde{K} \times \mathbb{R}}$ is countable-to-one.

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Observation 2

The inverse image of a point (u, v) is a line parallel to the xy-plane.

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Question

Does the generic $K \in \mathcal{C}$ contain uncountably many collinear points?
Does it contain 3 collinear points?

Thank you for your attention!

The main theorem again

Question (Tamás Keleti)

Is there a Besicovitch nullset $B \subseteq \mathbb{R}^2$ that meets every line not contained in it in a nullset?

Theorem (K, 2019)

There exists a Besicovitch set $B = \bigcup \mathcal{L}$ (where $\mathcal{L}$ is a family of lines) in the plane such that:

(1) B is closed;

(2) $\lambda^2(B) = 0$;

(3) for every line $e \notin \mathcal{L}$ we have $\lambda^1(B \cap e) = 0$;

(4) for every line $e \in \mathcal{L}$ we have $\lambda^1(e \cap \bigcup(\mathcal{L} \setminus \{e\})) = 0$.

Background

Theorem (K. Simon, B. Solomyak)

Let $K \subseteq \mathbb{R}^2$ be set with the following properties:

(1) it is self-similar;

(2) it has Hausdorff dimension 1;

(3) it satisfies the Open Set Condition;

(4) it is not on a line.

Then K is invisible from every point of the plane.