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Topological properties of trenched graphs

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Warsaw circle

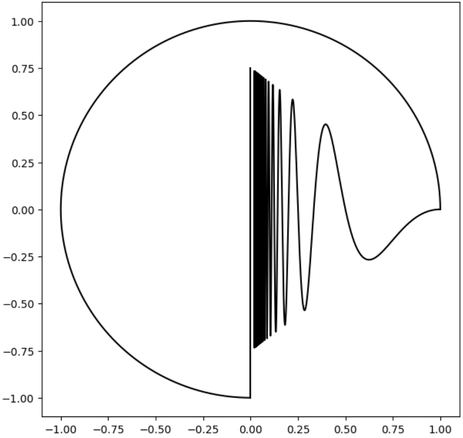

Figure: Warsaw Circle

Quasi-graphs[1]

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Definition

We say continuum X is *quasi-graph* if and only if there is a graph G and finite set $\{L_1, ..., L_n\}$ of pairwise disjoint oscillatory quasi-arcs in X such that:

1. (i) $X = G \cup \bigcup_{j=1}^{n} L_j$ and $End(X) \cup Br(X) \subset G$,
2. (ii) for each $0 \leq i \leq n$ $L_i \cap G = \{a_i\}$, where a_i is the endpoint of L_i,
3. (iii) $\omega(L_i) \subset G \cup \bigcup_{j=1}^{i-1} L_j$ for each $0 \leq i \leq n$,
4. (iv) if $\omega(L_i) \cap L_j \neq \emptyset$ for some $0 \leq i, j \leq n$, then $\omega(L_i) \supset L_j$

[1] E. Shi J. Mai. "Structures of quasi-graphs and ω-limit sets of quasi-graph maps". In: *Trans. Amer. Math. Soc.* 369.1 (2017), pp. 139–165. ISSN: 0002-9947. DOI: 10.1090/tran/6627. URL: https://doi.org/10.1090/tran/6627.

Warsaw circle as a quasi-graph

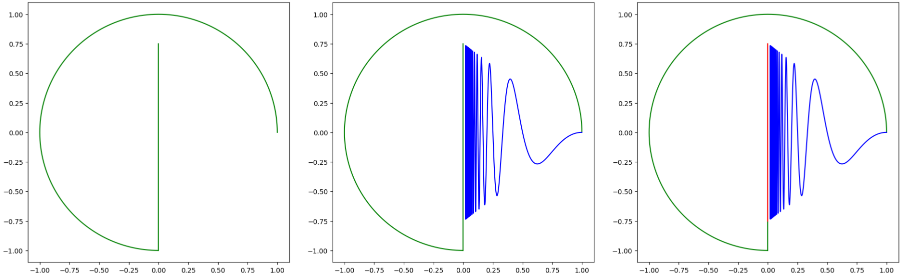

Figure: Construction of the Warsaw Circle as a quasi-graph

Generalized sin(1/x)-type continua[2]

Definition

A continuum X is a *generalized sin(1/x)-type continuum* if there exists a topological graph Y and a monotone map $\phi\colon X \to Y$ with following properties:

1. $\phi^{-1}(y)$ is nowhere dense in X for any $y \in Y$,
2. $\phi^{-1}(D)$ is dense in X, where $D = \{y \in Y$ such that $\phi^{-1}(y)$ is degenerate$\}$,
3. if Y_0 is a subcontinuum of $\phi^{-1}(y)$ and $\epsilon > 0$ then there exists an arc $[a, b] \subset Y$ such that $d_H(Y_0, \phi^{-1}([a, b])) < \epsilon$

Sets $\phi^{-1}(y)$ are called *fibers* of X. Nondegenerete fibers are called *trenches* of X.

[2] C. Mouron L. Hoehn. "Hierarchies of chaotic maps on continua". In: *Ergodic Theory Dynam. Systems* 34.6 (2014), pp. 1897–1913. ISSN: 0143-3857. DOI: 10.1017/etds.2013.32. URL: https://doi.org/10.1017/etds.2013.32.

Warsaw circle as generalized sin(1/x)-type continuum

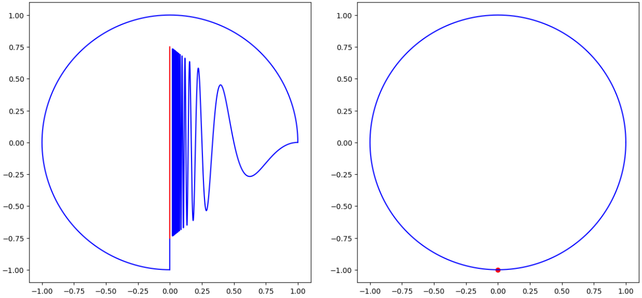

Figure: Warsaw Circle and its image under monotone mapping ϕ from definiton of sin(1/x)-type continuum

Arc limiting a triod

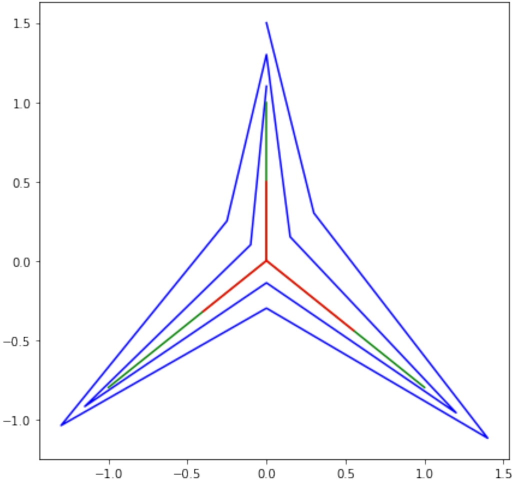

Figure: Quasi-arc with triod as the limit set.

Quasi-graph thats not a sin(1/x)-type continuum

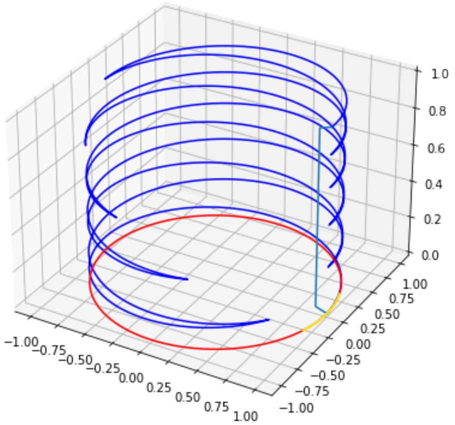

Figure: A quasi-graph whose limit set is circle, but is not a generalized sin(1/x)-type continuum

Trenched graphs

Definition

We will call continuum X a *trenched graph* if there is a topological graph Y and monotone map $\phi\colon X \to Y$ with following properties:

(i) $\phi^{-1}(y)$ is nowhere dense in X for all $y \in Y$,

(ii) $\phi^{-1}(D)$ is dense in X, where $D = \{y \in Y$ such that $\phi^{-1}(y)$ is degenerate$\}$,

As with the generalized sin(1/x)-type continua, we will call sets $\phi^{-1}(y)$ *fibers* of X. Similarly, nondegenerete fibers will be called *trenches* of X.

Definition

We say trenched graph has regular fibers if all fibers $\phi^{-1}(y)$ are either a singleton or a trenched graph.
We say trenched graph is regular if any subcontinuum $Y \subset X$ is singleton or a trenched graph with regular fibers.

Sufficient conditon for being sin(1/x)-type continuum

Lemma

Let X be a quasi-graph. Then X is a regular trenched graph with mapping $\phi\colon X \to X/_\sim$, where relation $\sim$ collapses connected components of limit sets and ϕ is a natural projection.

Theorem

Let $X = G \cup \bigcup_{j=1}^{n} L_j$ be a quasi-graph. Assume that for every connected component $\Lambda \subset \bigcup \omega(L_j)$ the following assertions hold:

1. *There is a quasi-arc L in X such that $\omega(L) = \Lambda$ and*
2. *Continuum Λ is arc-like.*

Then X is a generalized sin(1/x)-type continuum.

Sin(1/x)-type continuum with branching point in a trench

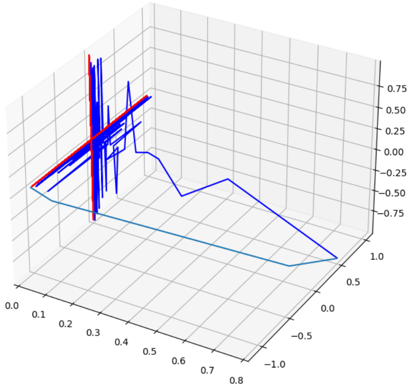

Figure: A quasi-graph which is generalized sin(1/x)-type continuum and contains 4-star as a trench

Necessary conditon for being sin(1/x)-type continuum

Theorem

Let X be a quasi-graph that is a generalized sin(1/x) type continuum. Then for every connected component $\Lambda \subset \bigcup \omega(L_j)$ there is a quasi-arc $L \subset X$ such that $\omega(L) = \Lambda$

Set of trenches doesn't need to be closed

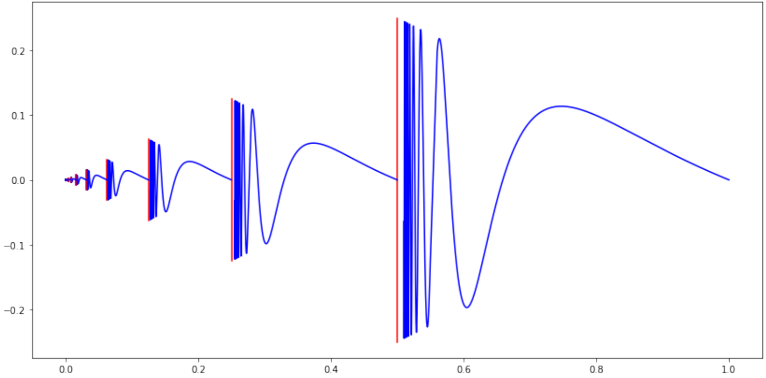

Figure: Generalized sin(1/x)-type continuum whose set of trenches is not closed

There can be infinite hierarchy of quasi-arcs

$$A = \bigcup_{n=0}^{\infty} \sigma^n(\{x, f(x), \ldots, f^n(x), \ldots) : x \in (0,1]\}) \cup \{0\}^\infty. \qquad (1)$$

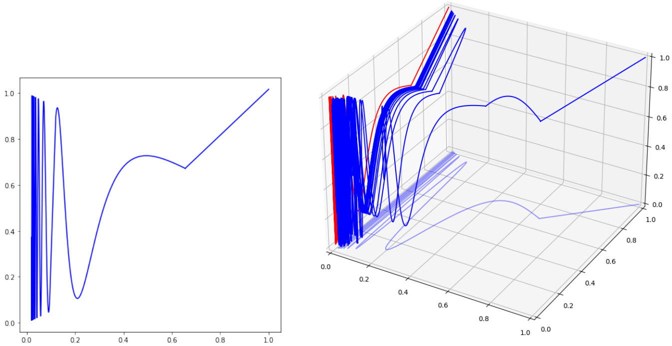

Figure: $f : (0,1] \to (0,1]$ and continuum of order 2

Double-sided sin(1/x)-continuum

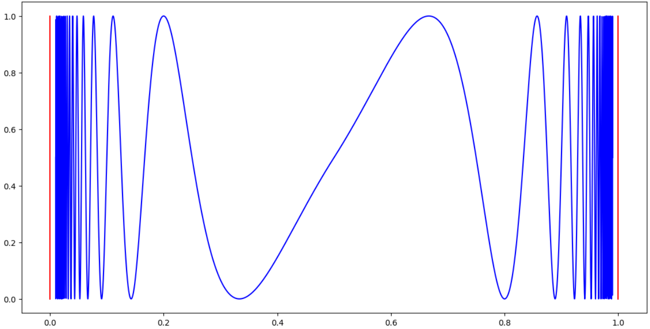

Figure: Continuum X

A strange sin(1/x)-type continuum

$$\widehat{X} = \{(x_0, x_1, \ldots, x_n, \ldots) : x_0 \in [0,1], \forall i \quad (x_{i-1}, x_i) \in X\}$$

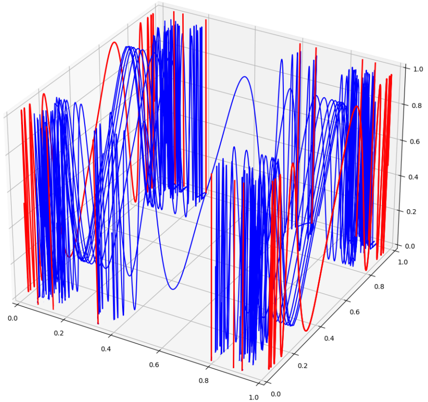

Figure: Projection of $\widehat{X}$ onto three dimensional space

Sufficient and necessary condition to be a quasi-graph.

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We can prove that:

1. $\widehat{X}$ is a sin(1/x)-type continuum,
2. Set of trenches of $\widehat{X}$ is dense in $\widehat{X}$,
3. Every fiber is either a singleton or is homeomorphic to $\widehat{X}$,
4. Continuum $\widehat{X}$ contains no arcs inside it,
5. Continuum $\widehat{X}$ is hereditarily decomposable.

Theorem

Let X be a trenched graph. Then X is a quasi-graph if and only if it is arcwise connected, regular and of finite order.