On generic topological embeddings

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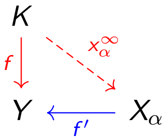
Let κ be a regular cardinal and $\mathfrak{K}$ be a full subcategory of a bigger category $\mathfrak{L}$ such that the following compatibility conditions are satisfied.

(L0) All $\mathfrak{L}$-arrows are epi.

(L1) Every inverse sequence of length κ in $\mathfrak{K}$ has the limit in $\mathfrak{L}$.

(L2) Every $\mathfrak{L}$-object is the limit of an inverse sequence in $\mathfrak{K}$.

(L3) For every inverse sequence $\vec{x}$ in $\mathfrak{K}$ with $K = \lim \vec{x}$ in $\mathfrak{L}$, for every $\mathfrak{K}$-object Y, for every $\mathfrak{L}$-arrow $f \colon K \to Y$ there exist $\alpha < \kappa$ and a $\mathfrak{K}$-arrow $f' \colon X_\alpha \to Y$ such that $f = f' \circ x_\alpha^\infty$.



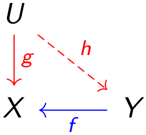
$\langle \mathfrak{K}, \mathfrak{L} \rangle$ is a free completion if (L0)-(L3) are satisfied.

$\mathfrak{K}$-generic

Now, an $\mathfrak{L}$-object U will be called $\mathfrak{K}$-generic if

(G1) $\mathfrak{L}(U, X) \neq \emptyset$ for every $X \in \mathrm{Obj}(\mathfrak{K})$.

(G2) For every $\mathfrak{K}$-arrow $f\colon Y \to X$, for every $\mathfrak{L}$-arrow $g\colon U \to X$ there exists an $\mathfrak{L}$-arrow $h\colon U \to Y$ such that $f \circ h = g$.



The Cantor set $\{0,1\}^\omega$

Let $\mathfrak{Fin}$ be a category of finite nonempty discrete spaces

as a full subcategory of the category

$\mathfrak{Comp}$ of compact metric spaces and continuous surjections

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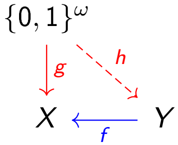
(G1) $\mathfrak{Comp}(\{0,1\}^\omega, X) \neq \emptyset$ for every $X \in \mathrm{Obj}(\mathfrak{Fin})$.

(G2) For every $\mathfrak{Fin}$-arrow $f\colon Y \to X$, for every $\mathfrak{Comp}$-arrow $g\colon \{0,1\}^\omega \to X$ there exists an $\mathfrak{Comp}$-arrow $h\colon \{0,1\}^\omega \to Y$ such that $f \circ h = g$.



The Cantor set is the $\mathfrak{Fin}$-generic object.

The Čech-Stone remainder ω^*

Let $\mathfrak{Comp}$ be a category of compact metric spaces and continuous surjections
and
let $\mathfrak{L}$ be a category of compact spaces of weight not greater than the continuum, with continuous surjections.

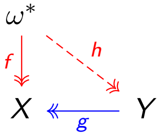
The following results of Parovičenko and Negrepontis imply that ω^* is $\mathfrak{Comp}$-generic.

(G1)Theorem (Parovičenko)

Every compact metric space is a continuous image of ω^*

(G2)Theorem (Negrepontis)

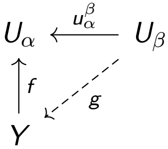
Assuming [CH], ω^* is compact Hausdorff space of weight ω_1 such that for every two continuous surjections $f\colon \omega^* \to X$ and $g\colon Y \to X$ with X and Y compact metric, there exists a continuous surjection $h\colon \omega^* \to Y$ such that $g \circ h = f$



A κ-Fraïssé sequence in $\mathfrak{K}$ is an inverse sequence $\vec{u}$ of regular length κ satisfying the following conditions:

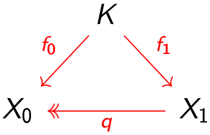
(U) For every object X of $\mathfrak{K}$ there exists $\alpha < \kappa$ such that $\mathfrak{K}(U_\alpha, X) \neq \emptyset$.

(A) For every $\alpha < \kappa$ and for every morphism $f\colon Y \to U_\alpha$, where $Y \in \mathrm{Obj}(\mathfrak{K})$, there exist $\beta \geq \alpha$ and $g\colon U_\beta \to Y$ such that $u_\alpha^\beta = f \circ g$.



Categories of continuous mappings, $\mathfrak{L}_K$, $\mathfrak{C}_K$

Fix a compact 0-dimensional space K such that $w(K) \leq 2^\omega$.
The objects of $\mathfrak{L}_K$ are continuous mappings $f: K \to X$, where X is compact 0-dimensional with $w(X) \leq \omega_1$.
Given two $\mathfrak{L}_K$-objects $f_0: K \to X_0$, $f_1: K \to X_1$, an $\mathfrak{L}_K$-arrow from f_1 to f_0 is a continuous surjection $q: X_1 \to X_0$ satisfying $q \circ f_1 = f_0$. The composition in $\mathfrak{L}_K$ is the usual composition of mappings.



Let $\mathfrak{C}_K$ be the full subcategory of $\mathfrak{L}_K$ whose objects are those $f: K \to X$ with $w(X) \leq \omega$.

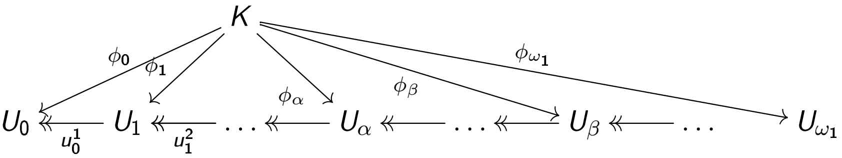
Theorem (Kubiś, K.,Turek)

Assume that $\vec{\phi} = (\phi_\alpha \colon \alpha < \omega_1)$ *is a continuous Fraïssé* ω_1*-sequence in* $\mathfrak{C}_K$*, where* $\phi_\alpha \colon K \to U_\alpha$ *for each* $\alpha < \omega_1$*. Then*

(1) $\vec{u} = (U_\alpha \colon \alpha < \omega_1)$ *is a Fraïssé sequence in the category* $\mathfrak{Comp}$*, whenever* K *is F-space.*

(2) *If* $\mathrm{w}(K) \leq \omega_1$*, then the limit map* $\phi_{\omega_1} \colon K \to \lim \vec{u}$ *has a left inverse i.e. there is* $r \colon \lim \vec{u} \to K$ *such that* $r \circ \phi_{\omega_1} = \mathrm{id}_K$*,*

(3) *The image* $\phi_{\omega_1}[K]$ *is a nowhere dense P-set in* $U_{\omega_1} = \lim \vec{u}$*.*



Corollary (Balcar, Frankiewicz, Mills)

Assuming CH, every compact 0-dimensional F-space of weight $\mathfrak{c}$ can be embedded as a nowhere dense closed P-set in ω^*.

- Let K be a compact 0-dimensional F-space of weight $\mathfrak{c}$.
- Assuming CH, there exists a continuous ω_1-Fraïssé sequence in $\mathfrak{C}_K$.
- $\vec{u} = (U_\alpha : \alpha < \omega_1)$ is a Fraïssé sequence in the category $\mathfrak{Comp}$.
- $U_{\omega_1} = \lim \vec{u}$ is $\mathfrak{Comp}$-generic object.
- By a characterization of Negrepontis, U_{ω_1} is homeomorphic to ω^*.
- The image $\phi_{\omega_1}[K]$ is a nowhere dense closed P-set in ω^*.

Theorem (Kubiś, K.,Turek)

Let $\eta\colon K \to \omega^*$ be an embedding such that $\eta[K]$ is a nowhere dense P-set of ω^* and K is a 0-dimensional compact space. Then $\eta\colon K \to \omega^*$ is $\mathfrak{C}_K$-generic.

Corollary (van Douwen, van Mill)

Assuming CH, every homeomorphism of nowhere dense closed P-sets in ω^* can be extended to an auto-homeomorphism of ω^*.

- By uniqueness of $\mathfrak{C}_K$-generic.

Corollary (van Douwen, van Mill)

Assuming CH, every nowhere dense closed P-set in ω^* is a retract of ω^*.

- Let $\eta\colon K \hookrightarrow \omega^*$ denote the inclusion map.
- We can represent $\eta\colon K \hookrightarrow \omega^*$ as the limit of a sequence $\vec{u} = \{\phi_\alpha : \alpha < \omega_1\}$ in the category $\mathfrak{C}_K$.
- $\eta\colon K \hookrightarrow \omega^*$ is $\mathfrak{C}_K$-generic.
- $\vec{u}$ is Fraïssé sequence in the category $\mathfrak{C}_K$.
- There is a retraction $r\colon \omega^* \to K$.

Categories of κ-ultrametric spaces

Let γ be an ordinal. A γ- ultrametric (also called an "inverse γ-metric") on a set X is a function $u\colon X \times X \to \gamma + 1$ such that for all $x, y, z \in X$:

(U1) $u(x, y) = \gamma$ if and only if $x = y$,

(U2) $u(y, z) \geq \min\{u(y, x), u(x, z)\}$ (ultrametric triangle law),

(U3) $u(x, y) = u(y, x)$ (symmetry).

Let κ, λ be infinite cardinals.
We define a λ-ultrametric $u\colon \kappa^\lambda \times \kappa^\lambda \to \lambda + 1$ by the formula:

$$u(a, b) = \sup\{\alpha < \lambda\colon a \restriction \alpha = b \restriction \alpha\}$$

for $a, b \in \kappa^\lambda$.

If X is a discrete space, then there is a natural ultrametric $d\colon X \times X \to 2$ on X, namely $d(a,b) = 1$ iff $a = b$ and $d(a,b) = 0$ iff $a \neq b$. Since $B_1(x) = \{x\}$ the set $\{x\}$ is open for any $x \in X$.
A closed ball of radius α and center x is a set of the form

$$B_\alpha(x) = \{y \in X : u(x,y) \geq \alpha\},$$

where $x \in X$ and $\alpha \in \gamma$. Each γ-ultrametric induces a topology whose base is the family of all closed balls of radius less then γ. We will call a space with this topology γ-ultrametric (or γ-metrizable).

Let λ be a regular cardinal. Fix a λ-ultrametric space (K, u) of weight κ. The objects of $\mathfrak{M}_K$ are uniformly continuous mappings $f\colon K \to X$, where (X, d) is a λ-ultrametric space of weight not greater than κ. Given two $\mathfrak{M}_K$-objects $f_0\colon K \to X_0$, $f_1\colon K \to X_1$, an $\mathfrak{M}_K$-arrow from f_1 to f_0 is a uniformly continuous surjection $q\colon X_1 \twoheadrightarrow X_0$ satisfying $q \circ f_1 = f_0$. The composition in $\mathfrak{M}_K$ is the usual composition of mappings. Let $\mathfrak{D}_K$ be the full subcategory of $\mathfrak{M}_K$ whose objects are those $f\colon K \to X$ where X is a discrete space of cardinality not greater than κ.
We say that a function $f\colon K \to X$ is uniformly continuous if

$$\forall_{\epsilon\in\tau}\exists_{\delta\in\kappa}\forall_{a,b\in K}\ u(a,b) \geq \delta \Rightarrow d(f(a), f(b)) \geq \epsilon,$$

where $u\colon K \times K \to \kappa + 1$ and $d\colon X \times X \to \tau + 1$ are ultrametrics.

λ is a regular cardinal.

Theorem (Kubiś, K., Turek)

Assume that $\vec{\phi} = (\phi_\alpha \colon \alpha < \lambda)$ is a continuous Fraïssé λ-sequence in $\mathfrak{D}_K$, where $\phi_\alpha \colon K \to U_\alpha$ for each $\alpha < \kappa$ and $\kappa^{<\lambda} = \kappa$. Then

1. *$\vec{u} = (U_\alpha \colon \alpha < \lambda)$ is a Fraïssé sequence in the category of discrete spaces of cardinality not greater than κ and uniformly continuous surjection.*
2. *The limit map $\phi_\lambda \colon K \to \lim \vec{u}$ has a left inverse, i.e. there is $r \colon \lim \vec{u} \to K$ such that $r \circ \phi_\lambda = \mathrm{id}_K$.*
3. *The image $\phi_\lambda[K] \subseteq U_\lambda = \lim \vec{u}$ is uniformly nowhere dense.*

We say that a subset A of a λ-ultrametric space (X, d) of weight κ is uniformly nowhere dense if for every $\alpha < \lambda$ there is $\beta > \alpha$ such that

$$\{B_\beta(a) \colon B_\beta(a) \cap A = \emptyset\, , a \in X\, , a \restriction \alpha = b \restriction \alpha\} \neq \emptyset$$

for every $b \in X$. Note that every uniformly nowhere dense subset of the ultrametric space κ^λ is nowhere dense.

λ is a regular cardinal

Theorem (Kubiś, K., Turek)

Assuming $\kappa^{<\lambda} = \kappa$, there exists a continuous Fraïssé λ-sequence in $\mathfrak{D}_K$.

Corollary (Kubiś, K., Turek)

Assuming $\kappa^{<\lambda} = \kappa$, a λ-ultrametric space (K, u) of weight κ can be uniformly embedded into κ^λ as a uniformly nowhere dense subset.

Theorem (Kubiś, K., Turek)

If $\eta \colon K \to \kappa^\lambda$ is uniformly embedded such that $\eta[K]$ is uniformly nowhere dense in the λ-ultrametric space κ^λ, then $\eta \colon K \to \kappa^\lambda$ is $\mathfrak{D}_K$-generic.

λ is a regular cardinal

Corollary (Kubiś, K.,Turek)

Assuming $\kappa^{<\lambda} = \kappa$, *every uniform homeomorphism of uniformly nowhere dense sets in* κ^λ *can be extended to a uniform auto-homeomorphism of* κ^λ.

Corollary (Kubiś, K.,Turek)

Assuming $\kappa^{<\lambda} = \kappa$, *every uniformly nowhere dense set in* κ^λ *is a uniform retract of* κ^λ.

Refereces

W. Kubiś, A. Kucharski and S. Turek, *On generic topological embeddings*, preprint, arXiv:2310.05043, (2023)

Thank You for Your attention!