A generic Banach space and a generic operator

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The talk is based on a result from the paper

- M. Cúth, M. Doležal, M. Doucha and O. Kurka, *Polish spaces of Banach spaces*, Forum Math. Sigma **10**, Paper No. e26, 28 p. (2022).

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By the *Gurariĭ space* we mean the (up to isometry) only separable Banach space $\mathbb{G}$ with the property that for every $\varepsilon > 0$, every finite-dimensional Banach spaces X and Y with $X \subseteq Y$ and every linear isometry $f : X \to \mathbb{G}$, there exists an ε-isomorphic embedding $g : Y \to \mathbb{G}$ extending f.

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- first constructed by V. I. Gurariĭ (1966)
- uniqueness up to isometry proved by W. Lusky (1976)
- quite simple proof of uniqueness by W. Kubiś and S. Solecki (2013)
- $\mathbb{G}$ can be obtained as a Fraïssé limit (I. Ben Yaacov?, W. Kubiś?)

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For $\mu \in \mathcal{P}$, let X_μ denote the completion of the quotient $(c_{00}, \mu)/N_\mu$, where $N_\mu = \{x \in c_{00} : \mu(x) = 0\}$.
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Remark

The motivation for introducing this coding of separable Banach spaces is that the “classical” coding due to B. Bossard is only a standard Borel spaces.
(This was partially resolved by G. Godefroy and J. Saint-Raymond.)

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The set of $\mu \in \mathcal{P}$ *such that* X_μ *is isometric to* $\mathbb{G}$ *is a dense* G_δ*-subset of* $\mathcal{P}$.

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Other results from our paper:

- the isometry class of an inf.-dim. space is closed if and only if it is isometric to ℓ_2
- the isomorphism class of an inf.-dim. space is an F_σ-set if and only if it is isomorphic to ℓ_2
- the isometry class of $L_p, p \neq 2, p \neq \infty$, is a G_δ-complete set
- the isometry class of $\ell_p, p \neq 2, p \neq \infty$, and of c_0 is an $F_{\sigma\delta}$-complete set

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Remark

The spaces whose isometry classes are G_δ-sets, so-called guarded Fraïssé spaces, are investigated in a forthcoming paper by M. Cúth, N. de Rancourt and M. Doucha.

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- constructed as a Fraïssé limit by J. Garbulińska-Węgrzyn and W. Kubiś (2015)
- they also proved that it is unique up to isometry
- T. Banakh and J. Garbulińska-Węgrzyn proved that it forms a dense G_δ-set in the space $\mathcal{B}(\mathbb{G})$ of non-expansive linear operators on $\mathbb{G}$ with the strong operator topology

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Remark

A coding of all operators based on Bossard's approach was introduced by K. Beanland and D. Freeman.
However, we do not consider the same class of objects, as our operators $T : X \to Y$ are non-expansive and fulfill $\overline{T(X)} = Y$.

Conjecture

The set of $(\mu, \nu) \in \mathcal{O}$ such that $T_{\mu,\nu}$ is a Gurariĭ operator is a dense G_δ-subset of $\mathcal{O}$.