On topological Ramsey spaces over and around Fraïssé limits

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(Joint work with Natasha Dobrinen[2])

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Thanks

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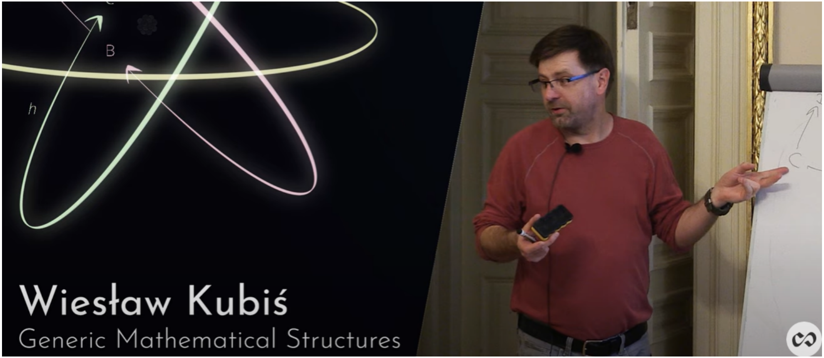
Thanks



Outline of the talk

1 Topological Ramsey spaces

2 Three constructions

3 Concluding meditations

Next . . .

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2 Three constructions

3 Concluding meditations

Finite, infinite and very infinite Ramsey statements

$$(\forall k)(\forall n)(\exists L)\ L \longrightarrow (n)^k_2 \quad \text{Finite Ramsey Theorem}$$

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- Easy construction of a bad coloring (AC)
- Problem: Too many colorings of $\omega^{[\infty]}$
- Idea: *Consider special colorings!*

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Baire sets?

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- metric topology is not rich enough
- *Refine the topology!*

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Topologize $\omega^{[\infty]}$ by the exponential = Vietoris = *Ellentuck topology*

Basic open sets ($a \in \omega^{[<\infty]}$, $B \in \omega^{[\infty]}$):

$$[a, B] = \{X \in \omega^{[\infty]} : a \sqsubset X \wedge X \subseteq a \cup B\}$$

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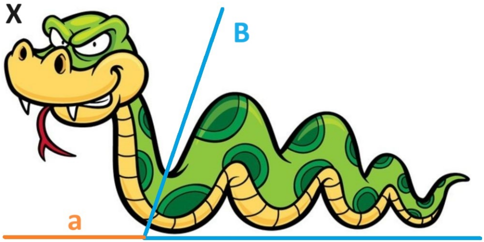

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Theorem (Ellentuck 1974) Every Baire $\mathcal{A} \subseteq \omega^{[\infty]}$ is Ramsey w.r.t. the exponential topology on $\omega^{[\infty]}$.

Baire hunt

With $\mathcal{X} \subseteq \omega^{[\infty]}$:

- $\mathcal{X}$ is completely Ramsey if for every basic open set $[a, B]$ there is a $C \in [a, B]$ such that either $[a, C] \subseteq \mathcal{X}$ or $[a, C] \cap \mathcal{X} = \varnothing$;
- $\mathcal{X}$ is Ramsey null if for every basic open set $[a, B]$ there is a $C \in [a, B]$ such that $[a, C] \cap \mathcal{X} = \varnothing$.

Theorem ($\Leftrightarrow$ Ellentuck) Consider $\omega^{[\infty]}$ with the exponential topology and let $\mathcal{X} \subseteq \omega^{[\infty]}$.

(a) $\mathcal{X}$ is Baire iff it is completely Ramsey;

(b) $\mathcal{X}$ is meager iff it is Ramsey null.

Topological Ramsey spaces

Topological Ramsey space $\rightarrow$ abstraction of the Ellentuck space

Principal references:

1. T. J. Carlson. *Some unifying principles in Ramsey theory.* Discrete mathematics 68 (1988), 117–169.
2. S. Todorcevic. *Introduction to Ramsey spaces.* Annals of Mathematics Studies 174, Princeton University Press 2010.

Topological Ramsey spaces

An approximation space is a triple $(\mathcal{R}, \leqslant, r)$ where:

- $\leqslant$ is a preorder on $\mathcal{R}$,
- $r : \omega \times \mathcal{R} \to \mathcal{AR}$ (written $r_n(A)$ instead of $r(n, A)$).

A1. (Sequencing)

1. $r_0(A) = \varnothing$ for all A;
2. if $A \neq B$ then $r_n(A) \neq r_n(B)$ for some n;
3. if $r_n(A) = r_m(B)$ then $m = n$ and $r_k(A) = r_k(B)$ for all $k < n$.

For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$:

$$[a, B] = \{X \in \mathcal{R} : X \leqslant B \wedge (\exists n) r_n(X) = a\}$$

These are basic open sets of the Ellentuck topology on $\mathcal{R}$.

Topological Ramsey spaces

With $\mathcal{X} \subseteq \mathcal{R}$:

- $\mathcal{X}$ is (completely) Ramsey if for every basic open set $[a, B] \neq \varnothing$ there is a $C \in [a, B]$ such that either $[a, C] \subseteq \mathcal{X}$ or $[a, C] \cap \mathcal{X} = \varnothing$;
- $\mathcal{X}$ is Ramsey null if for every basic open set $[a, B] \neq \varnothing$ there is a $C \in [a, B]$ such that $[a, C] \cap \mathcal{X} = \varnothing$.

Definition. An approximation space $(\mathcal{R}, \leqslant, r)$ is a topological Ramsey space if every Baire set is Ramsey and every meager set is Ramsey null w.r.t. the Ellentuck topology.

Topological Ramsey spaces

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Topological Ramsey spaces

A1. (Sequencing) ...

A2. (Finitization) There is a quasiordering $\leqslant_{\text{fin}}$ on $\mathcal{AR}$ such that:

1. $\{a \in \mathcal{AR} : a \leqslant_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$;
2. $A \leqslant B$ if $(\forall n)(\exists m) r_n(A) \leqslant_{\text{fin}} r_m(B)$;
3. for all $a, b, c \in \mathcal{AR}$, if $a \sqsubset b$ and $b \leqslant_{\text{fin}} c$ then there is a $d \in \mathcal{AR}$ such that $a \leqslant_{\text{fin}} d \sqsubset c$.

Topological Ramsey spaces

A1. (Sequencing) ...

A2. (Finitization) ...

A3. (Amalgamation) Let $a \in \mathcal{AR}$, $B \in \mathcal{R}$ and let $\mathrm{depth}_B(a) = n$.

1. $[a, C] \neq \varnothing$ for all $C \in [n, B]$.
2. If $C \in \mathcal{R}$ such that $C \leqslant B$ and $[a, C] \neq \varnothing$ then there is a $D \in [n, B]$ such that $\varnothing \neq [a, D] \subseteq [a, C]$.

Topological Ramsey spaces

A1. (Sequencing) ...

A2. (Finitization) ...

A3. (Amalgamation) ...

A4. (Pigeonhole) Let $a \in \mathcal{AR}_k$, let $B \in \mathcal{R}$ such that $\text{depth}_B(a) = n$ and let $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then there is a $C \in [n, B]$ such that $r_{k+1}[a, C] \subseteq \mathcal{O}$ or $r_{k+1}[a, C] \subseteq \mathcal{O}^c$.

Topological Ramsey spaces

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Abstract Ellentuck Theorem (Carlson 1988) Let $(\mathcal{R}, \leqslant, r)$ be an approximation space closed in the metric topology.
If $(\mathcal{R}, \leqslant, r)$ satisfies **A1–A4** then it is a top Ramsey space.

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Spectacular applications in Ramsey theory, set theory (forcing), Banach spaces, ...

Structural Ramsey theory

Generalize Ramsey-type results to first-order relational structures

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Ellentuck Theorem	$\rightarrow$	**?** (∞-dim struct Ramsey th?)

Towards ∞-dimensional structural Ramsey theory

Not much is known:

- Dobrinen, Mijares, Trujillo 2017: *Topological Ramsey spaces from Fraïssé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points*
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For a Fraïssé limit $\mathcal{F}$:

Goal 1: Construct a topological Ramsey space $\mathcal{R} \subseteq \binom{\mathcal{F}}{\mathcal{F}}$.

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Goal 2: Infer properties of $\mathcal{F}$ using the machinery of topological Ramsey spaces.

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Do *as much as possible* in the language of category theory.

axiomatic approach $\rightarrow$ general notions

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$\rightarrow$ automatic dualization

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Unfortunately, we shall have to scale back to the language of relational structures very quickly.

Categorical setup

Assumptions on C:

- C is locally small: all $\hom_{\mathrm{C}}(A, B)$ are sets;
- C is directed: for all A, B there is a C such that $A \to C \leftarrow B$;
- morphisms are mono: if $f \cdot g = f \cdot h$ then $g = h$.
- Ramsey property: for all A, B there is a C such that $C \longrightarrow (B)^A_2$.
- $C \longrightarrow (B)^A_2$: for every coloring $\chi : \hom_{\mathrm{C}}(A, C) \to \{0, 1\}$ there is a $w \in \hom_{\mathrm{C}}(B, C)$ s.t. $|w \cdot \hom_{\mathrm{C}}(A, B)| = 1$.

Categorical setup

A category C is a Ramsey category of finite objects if:

- C is a directed category whose morphisms are mono;
- C has the Ramsey property;
- the skeleton S of C has at most countably many objects;
- for every $S \in \mathrm{Ob}(\mathsf{S})$ there are only finitely many morphisms in S whose codomain is S.

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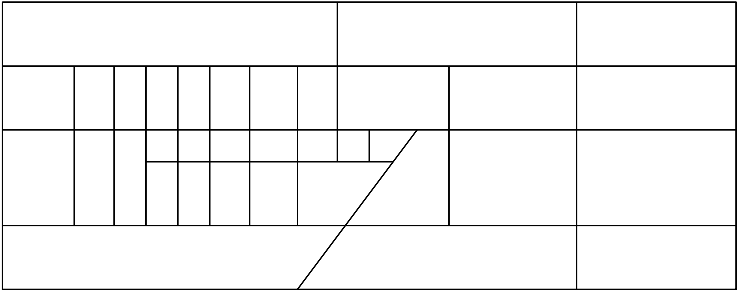

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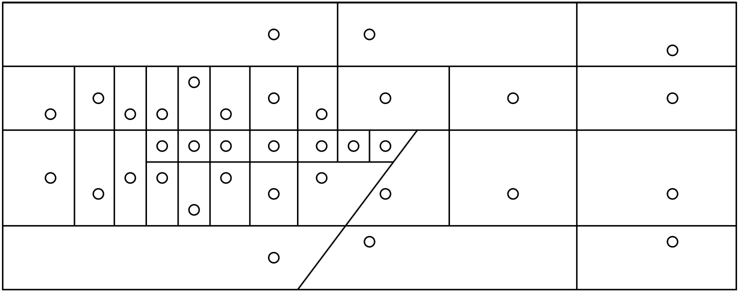

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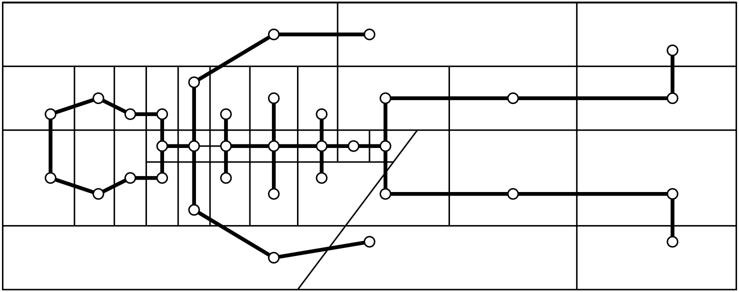

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A category is skeletal if it coincides with its skeleton.

NB. A category and its skeleton have the same Ramsey-related properties (def's invariant under isomorphism)

Construction 1: Pincushions

Generalizes

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Setup:

- A skeletal Ramsey category of finite objects C

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Setup:

- A Fraïssé sequence $Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$ in C

C

$$Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$$

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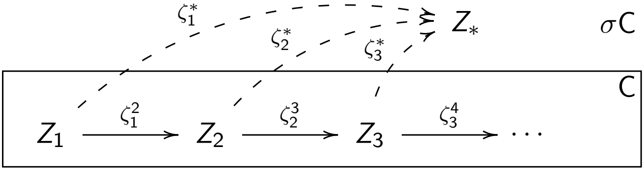
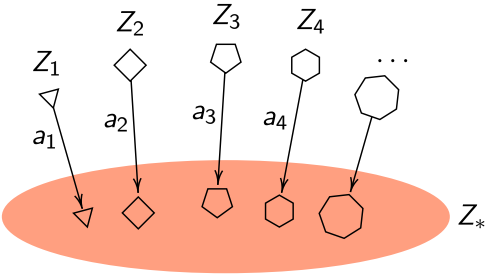
Setup:

- Its Fraïssé limit Z_* in some ambient category σC together with $\zeta_n^* : Z_n \to Z_*$, $n \in \mathbb{N}$

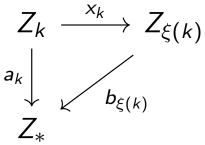


Construction 1: Pincushions

- $\mathcal{R} = \{\text{all sequences of morphisms } (a_i : Z_i \to Z_*)_{i\in\mathbb{N}}\}$



- $r_n\big((a_i)_{i\in\mathbb{N}}\big) = (a_1, \dots, a_n)$
- $(a_i)_{i\in\mathbb{N}} \leqslant (b_i)_{i\in\mathbb{N}}$ if there exists an increasing $\xi : \mathbb{N} \to \mathbb{N}$ and morphisms $x_n : Z_n \to Z_{\xi(n)}$ s.t. $a_k = b_{\xi(k)} \cdot x_k$



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Theorem. $(\mathcal{R}, \leqslant, r)$ is a topological Ramsey space.

Proof. Verify A1-A4.

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What makes us happy:

1. The construction is general *and dualizes*
2. Convenient setup for the general canonization theorem à la Pudlák-Rödl (*and its dual*)

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What makes us unhappy:

1. Not clear how pincushions relate to "copies of Z_* in Z_*"
2. Too big: many pincushions encode the same "copy of Z_*"

Construction 2: Distinguished copies

In a category of finite **relational structures and embeddings** refine Construction 1 so that pincushions correspond to certain ("distinguished") isomorphic copies of Z_*

Idea. Take any $(a_i)_{i\in\mathbb{N}} \in \mathcal{R}$:

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\end{array}
$$

A_* is a standard colimit of $A_1 \xrightarrow{\alpha_1^2} A_2 \xrightarrow{\alpha_2^3} A_3 \xrightarrow{\alpha_3^4} \cdots$

Write $A_* = \Theta\big((a_i)_{i \in \mathbb{N}}\big)$

Construction 2: Distinguished copies

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Implementation.

- A skeletal Ramsey category C of finite relational structures and embeddings

Construction 2: Distinguished copies

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Implementation.

- A Fraïssé sequence $Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$ in C

C

$$Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$$

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Implementation.

- ... which has the Hrushovski property (to be def'd soon!)

$$Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$$

C

Construction 2: Distinguished copies

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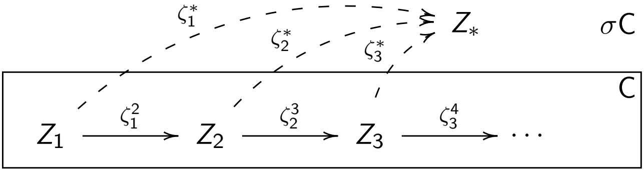
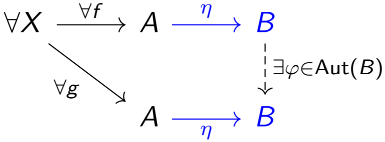
Implementation.

- ... and comes with a Fraïssé limit Z_* together with $\zeta_n^* : Z_n \to Z_*$, $n \in \mathbb{N}$



Hrushovski property (EPPA)

- Let C be a category, A, B objects of C and $\eta \in \hom_{C}(A, B)$. (η, B) is a Hrushovski pair for A if



- A sequence $Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$ has the Hrushovski property if (ζ_n^{n+1}, Z_{n+1}) is a Hrushovski pair for Z_n for all n.

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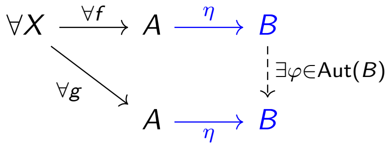

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Lemma. If $Z_1 \xrightarrow{\zeta_1^2} Z_2 \xrightarrow{\zeta_2^3} Z_3 \xrightarrow{\zeta_3^4} \cdots$ is universal for C and has the Hrushovski property then it is a Fraïssé sequence in C. Consequently, its colimit Z_* is ultrahomogeneous for C.

Construction 2: Distinguished copies

- $\overline{\mathcal{R}} = \{\Theta(\mathsf{a}) : \mathsf{a} \in \mathcal{R}\}$
- $\preccurlyeq$ and $\bar{r}$ defined somehow (*technical!*)

Theorem. $(\overline{\mathcal{R}}, \preccurlyeq, \bar{r})$ is a topological Ramsey space.

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1. $(\mathcal{R}, \preccurlyeq, r)$, $(\mathcal{R}', \preccurlyeq', r')$ ... approximation spaces
2. $\varphi : \mathcal{R} \to \mathcal{R}'$ is a homomorphism if $\varphi([n, A]_{\mathcal{R}}) = [n, \varphi(A)]_{\mathcal{R}'}$
3. Lemma (Carslon). If $\mathcal{R}$ is a top Ram spc and $\varphi : \mathcal{R} \to \mathcal{R}'$ a surjective homomorphism then $\mathcal{R}'$ is also a top Ram spc.
4. Θ is obviously surjective.
5. Hrushovski property $\Rightarrow$ Θ is a homomorphism

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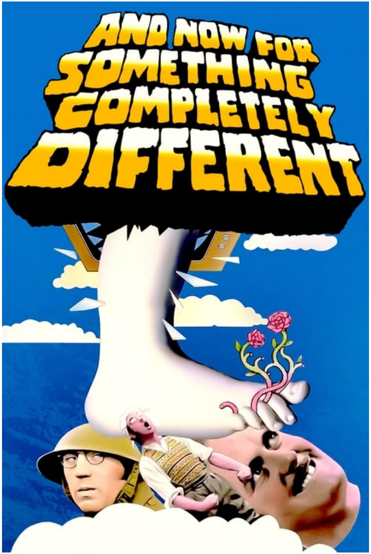
What makes us happy:

1. Elements of $\overline{\mathcal{R}}$ are structures isomorphic to Z_*

What makes us unhappy:

1. Still too big
2. $\overline{\mathcal{R}} \cap \binom{Z_*}{Z_*} = \varnothing$

Construction 3: Dense copies



Construction 3: Dense copies

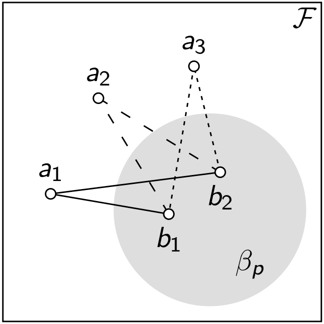
$\mathcal{F}$ ... a Fraïssé limit w/ SAP age

$\overline{a}$ ... a tuple in F^n

$p = p(x/\overline{a})$ ... a qf 1-type

$\beta_p = \{b \in F : \mathrm{qftp}_{\mathcal{F}}(b/\overline{a}) = p\}$

Enumerate all nonempty β_p's as
$\beta_1, \beta_2, \beta_3, \ldots$



Fact. The β_i's are basic open sets of a topology on F, call it $\tau^{\mathcal{F}}$.

Example. $\tau^{\mathbb{Q}}$ is the usual interval topology on $\mathbb{Q}$.

Construction 3: Dense copies

$\mathcal{F}$ ... a Fraïssé limit w/ SAP age

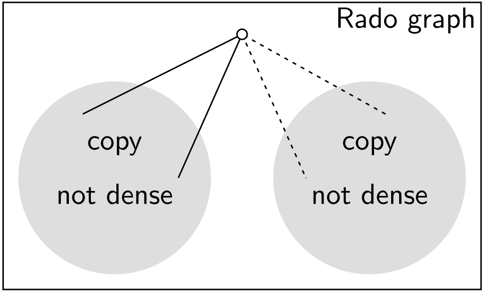
$\tau^{\mathcal{F}}$ ... topology generated by β_i's

Def. $D \subseteq F$ is a dense copy of $\mathcal{F}$ if D is dense w.r.t. $\tau^{\mathcal{F}}$.

$\mathcal{D}^{\mathcal{F}}$ all dense copies of $\mathcal{F}$.

Lemma. $\langle D \rangle_{\mathcal{F}} \cong \mathcal{F}$ for all $D \in \mathcal{D}^{\mathcal{F}}$.

Nonexample.



Construction 3: Dense copies

$\mathcal{F}$ ... a Fraïssé limit w/ SAP age

Enumerate F as $v_1 < v_2 < v_3 < \dots$

- $\mathcal{R}^{\mathcal{F}}$ is the set of all infinite subsets of $A \subseteq F$ such that

$$A = \{\underbrace{a_1^1}_{A(1)} < \underbrace{a_1^2 < a_2^2}_{A(2)} < \dots < \underbrace{a_1^n < a_2^n < \dots < a_n^n}_{A(n)} < \dots\}$$

  and $a_m^n \in \beta_m$ for all $m, n \in \mathbb{N}$ with $m \leqslant n$.

- $r_n^{\mathcal{F}}(A) = (A(1), A(2), \dots, A(n))$
- $A \sqsubseteq^{\mathcal{F}} B$ if there are $n_1 < n_2 < n_3 < \dots$ s.t. $A(i) \sqsubseteq B(n_i)$.

Construction 3: Dense copies

Theorem. $(\mathcal{R}^{\mathcal{F}}, \sqsubseteq^{\mathcal{F}}, r^{\mathcal{F}})$ is a topological Ramsey space.

Proof. Verify A1-A4.

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What makes us unhappy:

Lemma. Assume that there are $i \neq j$ s.t. $\beta_i \cap \beta_j = \varnothing$.
(NB: this is usually the case; fails for structures like $(\mathbb{N}, =)$.)
Then $\mathcal{R}^{\mathcal{F}}$ is nowhere dense in $\mathcal{D}^{\mathcal{F}}$.

Next . . .

1 Topological Ramsey spaces

2 Three constructions

3 Concluding meditations

Concluding meditations

Construction 1: Too big

Construction 2: Too big

Construction 3: Too small

Concluding meditations

Construction 1:	Too big
Construction 2:	Too big
Still missing:	*Just the right one!*
Construction 3:	Too small

Concluding meditations

Topological Ramsey spaces – modeled after the Ellentuck space

Concluding meditations

Topological Ramsey spaces – modeled after the Ellentuck space

- objects identified with sequences of finite approximations;

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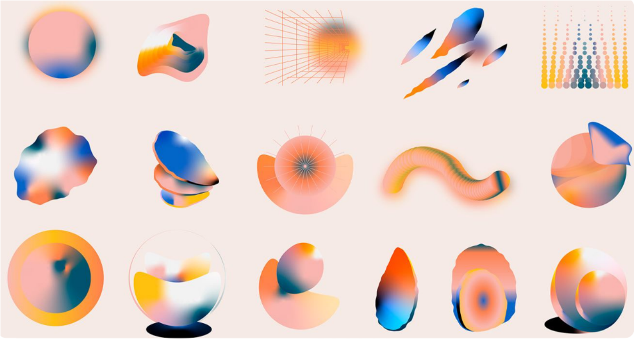

Concluding meditations

Topological Ramsey spaces – modeled after the Ellentuck space

- objects identified with sequences of finite approximations;
- all nth approximations are required to be isomorphic.

To fully understand ∞-dimensional structural Ramsey theory
we need a different theory of topological Ramsey spaces!