The Borel complexity of the bi-interpretability relation between omega-categorical structures

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Generic structures

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Results, and a question (will explain...)

Theorem (N, Schlicht and Tent, J. Math Logic 2021)

BI is Borel-below E_∞, where BI is the bi-interpretability relation between omega-categorical structures, and E_∞ is a Borel equivalence relation with all classes countable.

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Recall that $G \leq_c \mathrm{Sym}(\mathbb{N})$ is called oligomorphic if for each k, the action of G on $\mathbb{N}^k$ only has finitely many orbits.

Corollary

The topological isomorphism relation between oligomorphic groups is also Borel-below E_∞.

Question

Is there a lower bound other than $id_\mathbb{R}$ on the complexity?

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(a) Determine whether classes $\mathcal{C}$ of closed subgroups of S_∞ are Borel.

(b) If $\mathcal{C}$ is Borel, study the relative complexity of the topological isomorphism relation, using Borel reducibility $\leq_B$.

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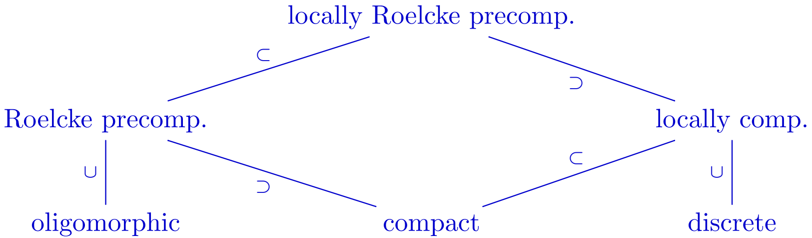
Let T_∞ be the undirected tree with each vertex of infinite degree.

- $\mathrm{Aut}(T_\infty)$ is locally R.p. (Zielinski), and not locally compact.
- The stabiliser of a vertex is Roelcke precompact.

Some Borel classes $\mathcal{C}$, and inclusion relations



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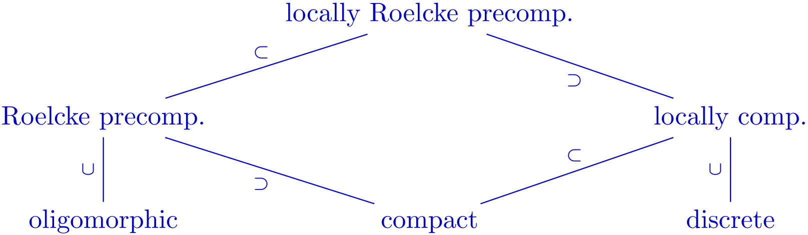

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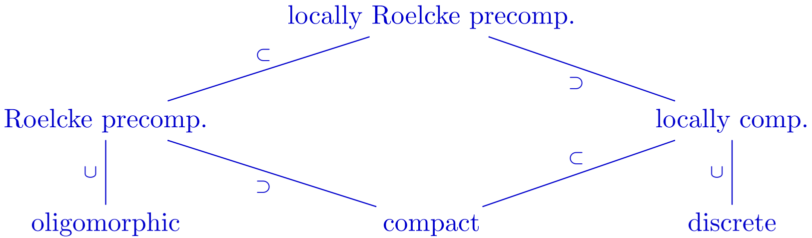

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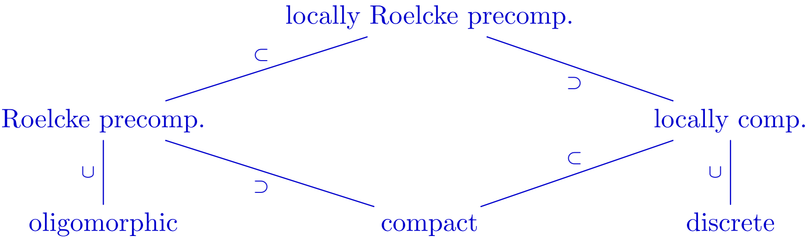

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- $\cong$ on the profinite groups is $\geq_B$ graph isomorphism (Kechris, N. and Tent, 2018).
- $\cong$ on the class of oligomorphic groups is $\leq_B$ a countable Borel equivalence relation (N., Schlicht and Tent, 2021).

Coarse group associated with a l.R.p. group

The following originiates in Kechris, N., Tent JSL 2018.

Given a locally R.p. G, let $\mathcal{M}(G)$ be its coarse group:

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Using descriptive set theory, we can view the operator $\mathcal{M}$ as a Borel function from locally R.p. groups to structures with domain $\mathbb{N}$.

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Definition

Given a structure $M \in \mathbf{CG}$, let $\mathcal{G}(M)$ be the closed subgroup of $\mathrm{Sym}(\mathbb{N})$ consisting of the permutations p such that

$$AB \sqsubseteq C \iff p(A)B \sqsubseteq p(C) \text{ for each } A, B, C \in M.$$

Recall that we have defined maps $\mathbf{LRP} \underset{\mathcal{G}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathbf{CG}$.

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Theorem (Borel duality for l.R.p. groups)

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  For each $G \in \mathbf{LRP}$ and each $M \in \mathbf{CG}$,
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As a consequence, for $G_0, G_1 \in \mathbf{LRP}$ and $M_0, M_1 \in \mathbf{CG}$, we have

$$G_0 \cong_{top} G_1 \iff \mathcal{M}(G_0) \cong \mathcal{M}(G_1)$$
$$M_0 \cong M_1 \iff \mathcal{G}(M_0) \cong_{top} \mathcal{G}(M_1)$$

The case of oligomorphic groups G

Note that G and hence each open subgroup is Roelcke precompact.

Theorem (NST, 21)

Among structures on $\mathbb{N}$ with a ternary relation symbol, let $\mathcal{D}$ be the closure under isomorphism of $\{\mathcal{M}(G) \colon G \text{ is oligomorphic}\}$.

(a) The class $\mathcal{D}$ is Borel.

(b) $\cong_{top}$ on the oligomorphic groups is Borel equivalent with the isomorphism relation on $\mathcal{D}$.

(a) is proved by a suitable axiomatisation of the class $\mathcal{D}$ using an infinitary language;
(b) is obtained via Borel duality for oligomorphic groups:

introduce a modification $\widehat{\mathcal{G}}$ of the "reverse" Borel operator $\mathcal{G}$ so that $\widehat{\mathcal{G}}(M)$ is oligomorphic for $M \in \mathcal{D}$.

A result of Hjorth and Kechris

The following will be used for showing that bi-interpretability on ω-categorical structures is $\leq_B E_\infty$ (a Borel equivalence relation with all classes countable):

Theorem (Hjorth and Kechris, APAL 1997, Th. 3.8)

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In our setting $\mathcal{D}$ is the class of coarse groups above.
We will verify the hypothesis on $\mathcal{D}$ by showing that the relation of bi-interpretability among ω-categorical structures is F_σ.