Big Ramsey degrees of the Urysohn sphere

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Big Ramsey degrees

An introduction in the discrete setting

Ramsey degrees: an overview

Theorem (Kechris–Pestov–Todorcevic 2004)

Let $\mathcal{F}$ be a well-behaved Fraïssé class and $\mathbb{F}$ its Fraïssé limit. The following are equivalent:

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The Ramsey property here is a version, in $\mathcal{F}$, of the finite Ramsey theorem:

Finite Ramsey theorem (1930)

For every $m \geqslant d \geqslant 1$ and every $k \geqslant 1$, there exists $n \geqslant m$ such that every colouring of $[n]^d$ with k color is monochromatic on a set of the form $[A]^d$, where $A \in [n]^m$.

(Here, $[A]^d$ is the set of d-elements subsets of A.)

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Theorem (Zucker 2016)

The following are equivalent:

- *$\mathcal{F}$ has finite small Ramsey degrees;*
- $\mathrm{Aut}(\mathbb{F})$ *has a metrizable universal minimal flow.*

In contrast, big Ramsey degrees are invariants measuring the default of existence of a version of the infinite Ramsey theorem.

Big Ramsey degrees: the example of $\mathbb{Q}$

Infinite Ramsey theorem (1930)

For every $d \geqslant 1$ and every colouring of $[\omega]^d$ with finitely many colours, there exists an infinite $X \subseteq \omega$ such that $[X]^d$ is monochromatic.

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Theorem (Galvin)

For every colouring $\psi\colon [\mathbb{Q}]^2 \to k$, where $k \in \omega$, there exists an order-copy $X \subseteq \mathbb{Q}$ of $\mathbb{Q}$ such that $[X]^2$ meets at most two colours.

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More generally, one can prove the existence of integers t_d such that for every $d \geqslant 1$, and every colouring of $[\mathbb{Q}]^d$ with finitely many colours, there exists an order-copy of $\mathbb{Q}$ meeting at most t_d many colours (Laver).

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The numbers t_d are called the big Ramsey degrees of $(\mathbb{Q}, <)$.

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Let X be an infinite discrete structure. Say that $A \in \mathrm{Age}(X)$ has finite big Ramsey degree in X if there exists an integer t_A such that every colouring on $\mathrm{Emb}(A, X)$ meets at most t_A colours on $\mathrm{Emb}(A, Y)$ for some $Y \in \binom{X}{X}$.

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The infinite Ramsey theorem exactly says that all the big Ramsey degrees on $(\omega, <)$ are equal to 1.

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Existence of finite big Ramsey degrees (sometimes with an explicit computation) has been proved for several classical discrete structures: the Rado graph (Sauer 2006, Laflamme–Sauer–Vuksanovic 2006), the universal homogeneous K_n-free graph (Dobrinen 2022, Balko–Chodounský–Dobrinen–Hubička–Koněcný–Vena–Zucker 2021+), the discrete Urysohn spehres (Balko–Chodounský–Hubička–Koněcný–Nešetřil–Vena 2021)...

Big Ramsey colourings

Suppose that $A \in \mathrm{Age}(X)$ has big Ramsey degree t_A in the structure X. As in the special case of Serpiński's colouring, one can easily prove the existence of a specific colouring $\chi \colon \mathrm{Emb}(A, X) \to t_A$ satisfiyng the two following properties:

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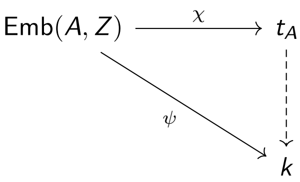

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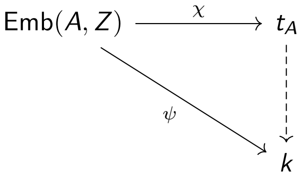

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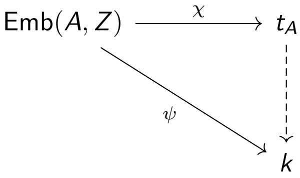

Moreover, t_A is the only number of colours for which a colouring with such properties exists. Call such a colouring a big Ramsey colouring.

The case of the Urysohn sphere

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The Urysohn sphere is ultrahomogeneous (in an exact sense, that is, every isometry between finite subsets of $\mathcal{U}_1$ extends to an onto isometry of $\mathcal{U}_1$), belongs to $\mathfrak{S}_1$, and contains isometric copies of all elements of $\mathfrak{S}_1^f$. It is characterized by those properties.

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$\mathcal{U}_1$ is the bounded version of the better-known Urysohn space, i.e. the Fraïssé limit of the class of all finite metric spaces.

Oscillation stability

The oscillation of a map $\chi\colon X \to Y$ between metric spaces, denoted by $\operatorname{osc}(\chi)$, is the diameter of its range, that is:

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From now on, letters $K, L, \ldots$ will denote compact metric spaces, and letters $\chi, \psi, \ldots$ will denote colourings, that are, 1-Lipschitz maps taking values in compact metric spaces.

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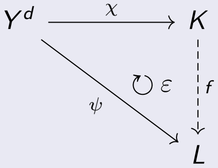

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Answer: K_1 is a singleton; this is a rephrasing of oscillation stability.

Description of the K_d's

From now on, all results should be taken with a grain of salt...

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Semimetric spaces

Definition

Let X be a set. A semimetric structure on X is a pair (M, m) of maps $X^2 \to [0, \infty)$ (the maximum distance and the minimum distance) satisfying the following axioms: for every $x, y, z \in X$,

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A semimetric space is a metric space endowed with a semimetric structure.

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More generally, if $A \subseteq Y$, one can put a semimetric structre on X by putting:

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If, in the latter example, one takes $Y \in \mathfrak{S}_1$, then for all $x, y \in X$ we have $m(x, y) + M(x, y) \leqslant 2$. Say that a semimetric structure is [0, 1]-valued if this condition is satisfied.

Maximal chains

Definition

Given two semimetric structures (M, m) and (M', m') on the same set X, say that (M', m') is more precise than (M, m), and write $(M, m) \leqslant (M', m')$, if for every $x, y \in X$, one has $M(x, y) \geqslant M'(x, y)$ and $m(x, y) \leqslant m'(x, y)$.

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The least $[0, 1]$-valued semimetric structure on a set X is the one for which we have $M(x, y) = 2$ and $m(x, y) = 0$ for all $x, y \in X$.

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Definition

Given a set X, denote by $\mathrm{SMC}(X)$ the set of all maximal chains of $[0, 1]$-valued semimetric structures on X.

Maximal chains

Example

Let $Y \in \mathfrak{S}_1$, $X \subseteq Y$, and $f : [0, +\infty[\to Y$ be a continuous fonction satisfying the two following conditions:

- for every $x, \in X$, $d(f(0), x) = 1$;
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For every $t \geqslant 0$ and $x, y \in X$, define:

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Then $(M_t, m_t)_{t\geqslant 0} \in \mathrm{SMC}(X)$.

Description of the K_d's

Given two semimetric structures (M, m) and (M', m') on the same set X, let

$$\delta((M,m),(M',m')) = \sup_{x,y\in X} \frac{1}{2}(|M(x,y)-M'(x,y)|+|m(x,y)-m'(x,y)|).$$

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Theorem (BdRHK)

$K_d = \mathrm{SMC}(d)$.

Thank you for your attention!