Generic one-dimensional maps and planar attractors

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This talk is in large part based on papers

This talk is based on two papers on attractors

1. J. Činč, P.O.,*Parametrized family of pseudo-arc attractors: physical measures and prime end rotations*, Proc. London Math. Soc., 2022
2. J. Činč, P.O.,*Parameterized family of annular homeomorphisms with pseudo-circle attractors*, arxiv, 2023

and in part on

3. J. Bobok, J. Činč, P.O., S. Troubetzkoy, *Periodic points and shadowing property for generic Lebesgue measure preserving interval maps*, Nonlinearity **35** (2022), Art. id: 2535.

Basic setting

1. X always compact metric space
2. $\#X > 1$ - nondegenerate
3. $T\colon X \to X$, always continuous
4. (X, T) - a dynamical system

Indecomposable arc-like continua

1. continuum C is arc-like if for every $\varepsilon > 0$ there is an ε-map $\pi \colon C \to I$ (i.e. $\mathrm{diam} \pi^{-1}(x) < \varepsilon$ for every $x \in I$).
2. circle-like, tree-like, graph-like are defined analogously.
3. continuum C is indecomposable if is **not** the union of two proper subcontinua.
4. hereditarily indecomposable if all nondegenerate subcontinua are indecomposable.
5. arc-like hereditarily indecomposable continuum is topologically unique - we call it the pseudoarc (Knaster; Moise; Bing).

The pseudo-arc

1. Bing **1951**: all three examples (of Knaster, Moise, and Bing) are homeomorphic

Theorem (Bing, 1951)

In $\mathbb{R}^n$ most of continua are pseudo-arcs (form a residual set in the space of all continua with Hausdorff metric).

Pseudo-circle

- (1951) R.H. Bing: pseudo-circle, a hereditarily indecomposable circle-like continuum that separates the plane into exactly two components. (uniqueness! Fearnley&Rogers)

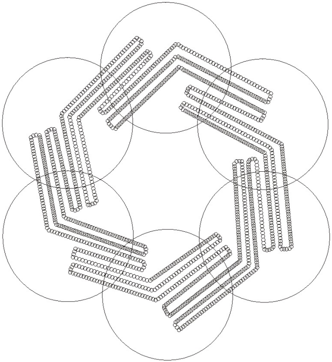


Figure: Construction by crooked circular chain (picture by Charatonik&Prajs&Pyrih)

Pseudo-circle in (smooth) dynamics

- Handel, 1982: Pseudo-circle as a minimal set of a C^∞ area-preserving planar diffeomorphism. Without the assumption of area-preserving as a planar attractor.
- Kennedy and Yorke, 1994: A C^∞ map on a 3-manifold with an invariant set consisting of uncountably many pseudo-circle components.
  Their example is stable under any small C^1 perturbation.
- Kennedy and Yorke 1995: A construction of a diffeomorphism with the same properties as above on an arbitrary 7-manifold.
- Boroński, O., 2015: A homeomorphism g on the 2-torus with the pseudo-circle Λ as a Birkhoff-like attractor[1].
- Béguin, Crovisier, Jäger, 2017: A decomposition of the 2-torus into pseudo-circles which is invariant under a torus homeomorphism semi-conjugate to an irrational rotation.

[1] the rotation set $\rho(g|\Lambda)$ is a nondegenerate interval.

- Handel, 1982: pseudo-circle as minimal set and attractor

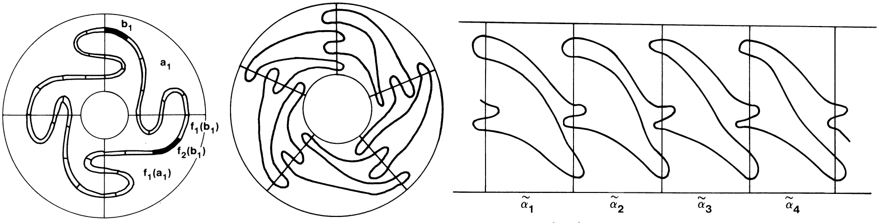


Figure: by M. Handel

Inverse limits

1. in general, inverse limit - $\varprojlim\{\{f_i\}_{i=0}^{\infty}, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f_i(x_{i+1}) = x_i\}$
2. we are interested in cases when there is one bonding map:

$$\mathbb{X} = \varprojlim\{f, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f(x_{i+1}) = x_i\}$$

3. shift homeo. - $\sigma_f(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$
4. f and σ_f share many dynamical properties, e.g.
   - dense periodic points
   - admissible periods of periodic points
   - $h_{top}(f) = h_{top}(\sigma_f)$
   - .....

Inverse limits and attractors

Theorem (Barge & Martin)

Every continuum $\mathbb{X} = \varprojlim\{f, [0,1]\}$, can be embedded into a disk D in such a way that

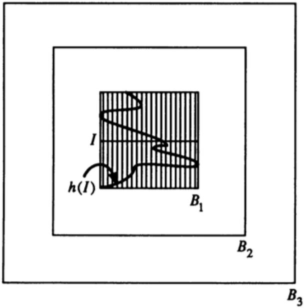
(i) $\mathbb{X}$ is an attractor of a homeomorphism $h\colon D \to D$,

(ii) $h|_{\mathbb{X}} = \sigma_f$; i.e. h restricted to $\mathbb{X}$ agrees with the shift homeomorphism induced by f, and

(iii) h is the identity on the boundary of D.

Remark

It was pointed by Barge & Roe that the same is true if f is a degree ± 1 circle map and h is an annulus homeomorphism.



M. Barge, J. Martin, *The construction of global attractors.* Proc. Amer. Math. Soc. **110** (1990), 523–525.

Boyland, de Carvalho, Hall (2016 BLMS, 2019 DCDS, 2021 G&T): Parametrized version of BBM's. Complete understanding of dynamics of BBM embeddings $\{\Lambda_s\}_{s\in[\sqrt{2},2]}$ of parametrized family of core tent maps $\{T_s\}_{s\in[\sqrt{2},2]}$ and their measure-theoretic properties. In particular, prime end rotation number of $\{\Lambda_s\}_{s\in[\sqrt{2},2]}$ varies continuously with s.

Inverse limits with natural measure - space $C_\lambda(I)$

1. λ – the Lebesgue measure on $I = [0, 1]$.
2. main space

$$C_\lambda(I) = \{f \in C(I);\ \forall\ A \subset [0, 1], A \text{ Borel}:\ \lambda(A) = \lambda(f^{-1}(A))\}.$$

3. we endow the set $C_\lambda(I)$ with the metric ρ of uniform convergence.
4. $(C_\lambda(I), \rho)$ is a complete metric space.

5. **a property P is typical in $(C_\lambda(I), \rho) \equiv$ the set of all maps with the property P is residual, maps bearing a typical property are called generic.**

J. Bobok, S. Troubetzkoy, *Typical properties of interval maps preserving the Lebesgue measure*, Nonlinearity **(**33)(2020), 6461–6501.

1. $C_\lambda(I)$ typical map
   1. is weakly mixing with respect to λ,
   2. is locally eventually onto,
   3. satisfies the periodic specification property,
   4. has infinite topological entropy,
   5. has its graph of Hausdorff dimension = lower Box dimension = 1.
2. and its graph upper Box dimension = 2
   [J. Schmeling, R. Winkler, *Typical dimension of the graph of certain functions*, Monatsh. Math. **119** (1995), 303–320].

Method of Minc and Transue

1. We say that $f \in C(I)$ is δ-crooked between a and b if,
   - for every two points $c, d \in I$ such that $f(c) = a$ and $f(d) = b$,
   - there is a point c' between c and d and there is a point d' between c' and d
   - such that $|b - f(c')| < \delta$ and $|a - f(d')| < \delta$.
2. We say that f is δ-crooked if it is δ-crooked between every pair of points.

Theorem

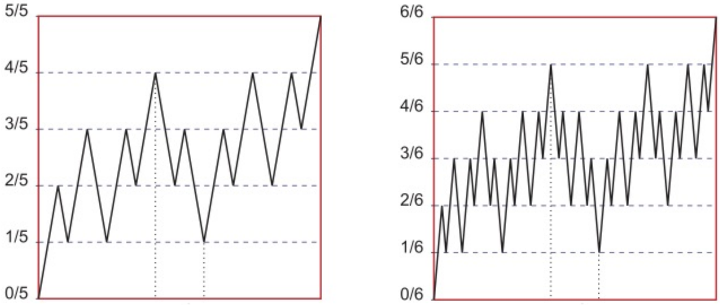
Let $f \in C(I)$ be a map with the property that,

- for every $\delta > 0$ there is an integer $n > 0$
- such that f^n is δ-crooked.

Then $\mathbb{X}$ is the pseudoarc.

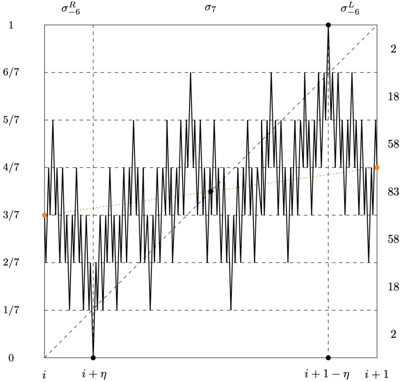
P. Minc and W. R. R. Transue, **A transitive map on $[0, 1]$ whose inverse limit is the pseudoarc**, Proc. Amer. Math. Soc. **111** (1991), 1165–1170.

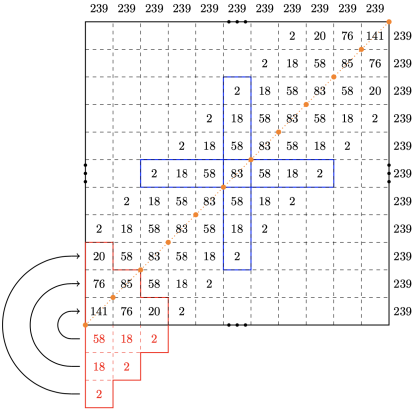
Construction of crooked map



- ε-crooked maps constructed by Minc and Transue.

Yet another typical property in $C_\lambda(I)$ (and $C_\lambda(\mathbb{S}^1)$)

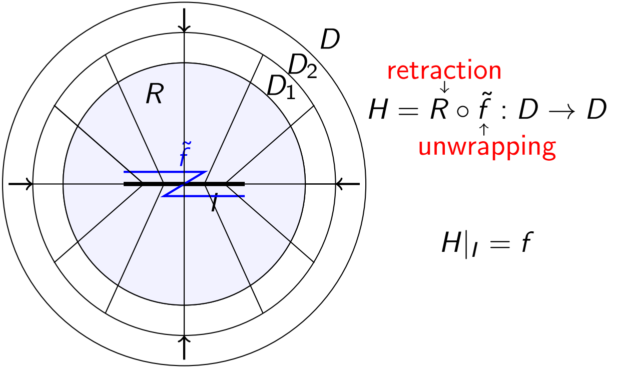




Theorem (Činč, O.)

For typical $f \in C_\lambda(I)$ the inverse limit $\varprojlim(I, f)$ is the pseudo-arc.

Brown-Barge-Martin construction of global attractors



$H|_I = f$

By Brown's approx. thm. $\hat{D} = \varprojlim(D, H)$ is a topological disk.
Furthermore, $\hat{I}$ is embedded in $\hat{D}$, $\hat{H}|_{\hat{I}} = \sigma_f : \hat{I} \to \hat{I}$.
Outcome: Homeomorphism of a topological disk with the unique global attractor homeomorphic to $\hat{I}$ and action on it conjugate to σ_f.

Finding generic class in planar attractors

Theorem

There exists a dense G_δ set of maps $A \subset \mathcal{T} \subset C_\lambda(I)$ and a parametrized family of homeomorphisms $\{\Phi_f\}_{f\in A} \subset \mathcal{H}(D,D)$ with Φ_f-invariant pseudo-arc attractors $\Lambda_f \subset D$ for every $f \in A$ so that

(a) $\Phi_f|_{\Lambda_f}$ *is topologically conjugate to* $\sigma_f : \hat{I}_f \to \hat{I}_f$.

(b) *The attractors* $\{\Lambda_f\}_{f\in A}$ *vary continuously in Hausdorff metric.*

...

(f) *The attractor Λ_f preserves induced weakly mixing measure μ_f invariant for Φ_f for any $f \in A$. Measures μ_f vary continuously in the weak* topology on $\mathcal{M}(D)$.*

Invariant measure for shift homeomorphism

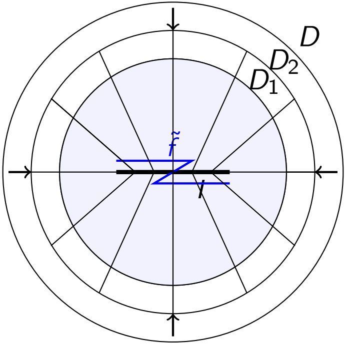
1. In what follows X is a compact Euclidean space with Lebesgue measure λ, and $f : X \to X$ be surjective and continuous.
2. Let μ be an f-invariant measure on $\mathcal{B}(X)$.
3. B_μ is a basin of μ for f if for all $g \in C(X)$ and $x \in B_\mu$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) = \int g d\mu.$$

4. We call measure μ physical for f if there exists a basin B_μ of μ for f and a Borel set B so that $B \subset B_\mu$ and $\lambda(B) > 0$.
5. An invariant measure ν for the natural extension $\sigma_f : \hat{X} \to \hat{X}$ is called inverse limit physical measure if ν has a basin B_ν so that $\lambda(\pi_0(B_\nu)) > 0$.

Lebesgue measure gives an advantage

Thm (Kennedy, Raines, Stockman 2010): If B is a basin of μ for f then $B_\nu := \pi_0^{-1}(B)$ is a basin of the measure ν induced by μ for σ_f (and vice versa). If μ is a physical measure for $f : X \to X$ then the induced measure ν on inverse limit $\hat{X}$ is an inverse limit physical measure for σ_f (and vice versa).



Dynamically equivalent embeddings

- Let X and Y be metric spaces. Suppose that
- $F : X \to X$ and $G : Y \to Y$ are homeomorphisms and
- $E : X \to Y$ is an embedding.

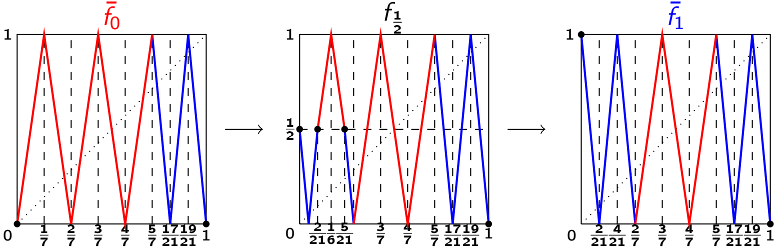
- If $E \circ F = G \circ E$ we say that the embedding E is a dynamical embedding of (X, F) into (Y, G).

- If E, resp. E', are dynamical embeddings of (X, F) resp. (X', F') into (Y, G), resp. (Y', G'), and
  - there is a homeomorphism $H : Y \to Y'$ so that $H(E(X)) = E'(X')$
  - which conjugates $G|_{E(X)}$ with $G'|_{E'(X')}$ we say that
  - the embeddings E and E' are dynamically equivalent.

A family $\{f_t\}_{t\in[0,1]} \subset \mathcal{T}$

For any $t \in I$ let $\bar{f}_t$ be defined $\bar{f}_t(\frac{2}{7}) = \bar{f}_t(\frac{4}{7}) = \bar{f}_t(\frac{17}{21}) = \bar{f}_t(1) = 0$ and $\bar{f}_t(\frac{3}{7}) = \bar{f}_t(\frac{5}{7}) = \bar{f}_t(\frac{19}{21}) = 1$ and piecewise linear between these points on the interval $[\frac{2}{7}, 1]$. Furthermore on interval $x \in [0, \frac{2}{7}]$ let:

$$\bar{f}_t(x) = \begin{cases} 7(x - t\frac{4}{21}); & x \in (1-t)[0, \frac{1}{7}] + t\frac{4}{21}, \\ 1 - 7(x - \frac{1}{7}(1-t) - t\frac{4}{21}); & x \in (1-t)[\frac{1}{7}, \frac{2}{7}] + t\frac{4}{21}, \\ \frac{21}{2}(x - t\frac{2}{21}); & x \in t[\frac{2}{21}, \frac{4}{21}], \\ 1 - \frac{21}{2}x; & x \in t[0, \frac{2}{21}], \\ \frac{21}{2}(x - \frac{2}{7}); & x \in [\frac{2}{7} - t\frac{2}{21}, \frac{2}{7}]. \end{cases}$$



Dynamically different embeddings

Theorem

There is a parametrized family of interval maps $\{f_t\}_{t\in[0,1]} \subset \mathfrak{T} \subset C_\lambda(I)$ *and a parametrized family of homeomorphisms* $\{\Phi_t\}_{t\in[0,1]} \subset \mathcal{H}(D,D)$ *with* Φ_t*-invariant pseudo-arc attractors* $\Lambda_t \subset D$ *for every* $t \in [0,1]$ *so that*

(a) $\Phi_t|_{\Lambda_t}$ *is topologically conjugate to* $\sigma_{f_t} : \hat{I}_{f_t} \to \hat{I}_{f_f}$.

(b) *The attractors* $\{\Lambda_t\}_{t\in[0,1]}$ *vary continuously in the Hausdorff metric.*

$\dots$

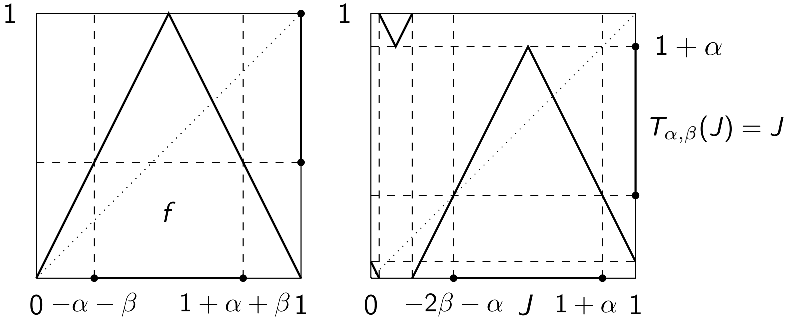
(d) *There are uncountably many dynamically non-equivalent planar embeddings of the pseudo-arc in the family* $\{(\Phi_t, \Lambda_t)\}_{t\in[0,1]}$.

Question

Are for every $t \neq t' \in [0,1]$ the attractors Λ_t and $\Lambda_{t'}$ (non)-equivalently embedded?

Example

Let f be a full tent map viewed as a circle map. For $\alpha < -\beta \pmod 1$ and $\alpha + \beta > -\frac{1}{2} \pmod 1$ we define the interval $J = J_{\alpha,\beta} := [-2\beta - \alpha, 1 + \alpha] \pmod 1$. We obtain that $T_{\alpha,\beta}(J) = J$. Therefore, $T_{\alpha,\beta}$ is not transitive for an open subset of $(\alpha, \beta) \in [0, 1)^2$.



1. A map $f : X \to X$ is locally eventually onto (leo), if for every open $U \subset X$ there exists $n \geq 1$ such that $f^n(U) = X$.
2. Bobok and Troubetzkoy (2020) showed the typical map f in $C_\lambda(I)$ is leo.

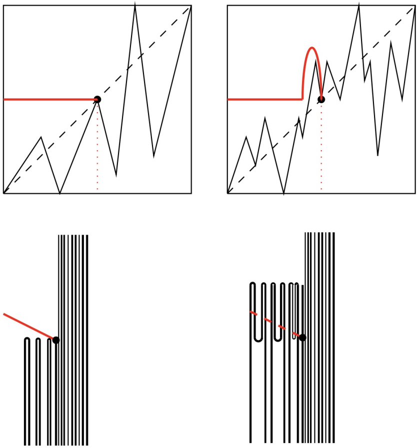
Recently, we were able to obtain a much stronger result in $C_\lambda(\mathbb{S}^1)$.

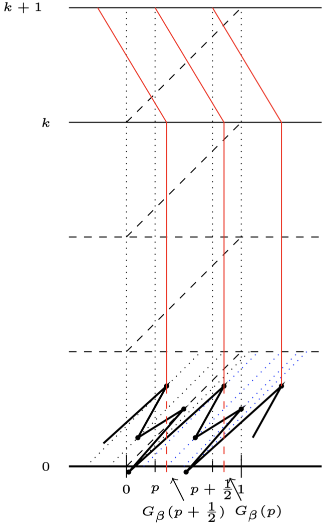
Theorem (Bobok, Činč, O., Troubetzkoy (2022))

There is an open dense set $O \subset C_\lambda(\mathbb{S}^1)$ such that:

1. each $f \in O$ is leo.
2. for each pair $\alpha, \beta \in [0, 1)$ and each $f \in O$ the map $T_{\alpha,\beta}(f)$ is leo.

Problems in circle maps





Theorem (Činč, O., 2023)

There is a parametrized family of circle maps $\{f_t\}_{t\in[0,1]} \subset \mathcal{T} \subset C_\lambda(\mathbb{S}^1)$ *so that every* f_t *is weakly mixing with respect to* λ *and a parametrized family of annular homeomorphisms* $\{\Phi_t\}_{t\in[0,1]} \subset \mathcal{H}(\mathbb{A},\mathbb{A})$ *varying continuously with* t *having* Φ_t*-invariant pseudo-circle cofrontier Birkhoff-like attractors* $\Lambda_t \subset \mathbb{A}$ *for every* $t \in [0,1]$ *so that*

(a) $\Phi_t|_{\Lambda_t}$ *is topologically conjugate to the natural extension* $\hat{f}_t \colon \hat{\mathbb{S}}^1 \to \hat{\mathbb{S}}^1$.

(b) Φ_t *is weakly mixing with respect to measure* $\hat{\mu}_t$ *induced by* λ. *Measures* $\hat{\mu}_t$ *vary continuously in weak* topology.*

(c) *The attractors* $\{\Lambda_t\}_{t\in[0,1]}$ *vary continuously in the Hausdorff metric.*

(d) *The outer prime ends rotation numbers of homeomorphisms* Φ_t *vary continuously with* t *in the interval* $[0,1/2]$.

(e) *There are uncountably many dynamically non-equivalent planar embeddings of the pseudo-circle in the family* $\{\Lambda_t\}_{t\in[0,1]}$.

Questions

Question 1: Does there exist an open dense set of leo volume preserving noninvertible maps that commute with rotations on higher-dimensional tori?

Question 2: What is the generic value of metric entropy for $C_\lambda(\mathbb{S}^1)$ or $C_\lambda(I)$?

Question 3: How to decide if two embedings using BBM are not equivalent. In particular, are our attractors non-equivalently embedded?

Question 4: Given any number $r \geq 0$, is there a transitive pseudo-arc map with entropy r?

Thank you!