Inner ultrahomogeneous groups

Tomasz Rzepecki

Uniwersytet Wrocławski

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Homogeneous groups

- $\mathbb{Q}$, all vector spaces (as pure groups),
- $\mathbb{Q}/\mathbb{Z}$, Prüfer groups,
- S_n — homogeneous if and only if $n \leq 3$,
- there is a very explicit classification of finite homogeneous groups (due to Cherlin and Felgner),
- Hall's universal group — the universal locally finite countable group.

Amalgamation of finite groups

- Hall defined a certain locally finite group Γ by a recursive construction, iterating the Cayley embedding $G \hookrightarrow S_G$.
- He showed that it has the following property: every isomorphism between finite subgroups of Γ is given by conjugation by some element of Γ.
- He used this to show that (in modern terminology) Γ is injective for finite groups and universal for the class of countable groups.
- By Fraïssé's theorem (contemporary to Hall's work) it follows that Γ is in fact a Fraïssé limit (now called Hall's universal group).
- Later, B.H. Neumann explicitly defined amalgamation for the class of finite groups using "permutation products".

Inner ultrahomogeneity

Definition

We say that a group Γ is *inner ultrahomogeneous* if every finite partial automorphism of Γ extends to an *inner* automorphism of Γ.

Remark

Given ultrahomogeneous Γ, inner ultrahomogeneity is equivalent to a condition about $\mathtt{Age}(\Gamma)$ which we call *inner EPPA*.

Definition

We say that a class $\mathcal{K}$ of groups has *inner EPPA* if for every $K \in \mathcal{K}$ and every finite partial automorphism p of K, there is $L \geq K$ in $\mathcal{K}$ such that p is extended by conjugation by some $\alpha \in L$.

Finite and finitely presentable groups

- The class of finite groups is a Fraïssé class and it has inner EPPA, its limit is Hall's universal group.
- The class of finitely generated groups has HP, JEP and AP and inner EPPA, but it is not countable (there are continuum many 2-generated groups).
- The class of finitely presentable groups is countable, has JEP, AP and inner EPPA.
- However, it does not have the hereditary property.
- On the other hand, every f.g. subgroup of a f.p. group is recursively presentable (the converse is also true).
- The class of f.g. recursively presentable groups is Fraïssé with inner EPPA.

Theorem (Hall, Song, Siniora, Rz., many parts probably folklore...)

If Γ is the Hall's universal group or the limit of finitely presentable groups, then:

1. Γ *is simple and divisible,*
2. Γ *is not $\aleph_0$-categorical and does not have q.e.,*
3. Γ *has ample generic automorphisms,*
4. *the formula $x \in C(y)$ has the independence property,*
5. *the formula $C(x) \subseteq C(y)$ has the strict order property,*
6. $x \in C(y_1) \setminus C(y_2)$ *has TP_2,*
7. Γ *has IP_n for each n, is straightly maximal...*

The goal is to understand how these properties can be derived from just inner ultrahomogeneity + some extra assumptions (e.g. there are three finite inner ultrahomogeneous groups which fail these, so we need to assume at the very least that Γ is infinite.).

Other examples

Some other examples of Fraïssé classes of groups with inner EPPA:

- S_1, S_2, S_3 are inner ultrahomogeneous (these are the only finite examples),
- given any countable transitive model $M \models \mathsf{ZFC}$, the class of finitely generated groups in M,
- any countable hereditary class of f.g. groups closed under amalgamated free products is a Fraïssé class, and if it is also closed under (finitary) HNN extensions, it has inner EPPA,
- in particular, we can start with any f.g. group (or countable class of groups) and close it under these operations,
- for example, if we start with a class of torsion-free groups, then as the limit, we obtain a torsion-free inner ultrahomogeneous group,
- it follows that every countable group is a subgroup of a countable inner ultrahomogeneous group (this can also be easily showed explicitly by a recursive construction).

Conjugacy and divisibility

Proposition

If Γ is an inner ultrahomogeneous group, then each conjugacy class in Γ consists of all elements of given order.

Proof.

If $g_1, g_2 \in \Gamma$ have the same order, them $g_1 \mapsto g_2$ is a partial automorphism. □

Proposition

If Γ is an ultrahomogeneous group, then an element $g \in \Gamma$ is n-divisible if and only if there is an element of order $n \cdot \mathrm{ord}(g)$.

Proof.

If h is of order $n \cdot \mathrm{ord}(g)$, then h^n is of order $\mathrm{ord}(g)$. Thus, we have $\sigma \in \mathrm{Aut}(\Gamma)$ with $\sigma(h^n) = g$. But then $\sigma(h)^n = g$. □

Simplicity, centre

Corollary

If Γ is an inner ultrahomogeneous group, then it is simple if and only if for each p prime or ∞, elements of order p either generate Γ or do not exist.

Proposition

The centre of an inner ultrahomogeneous group has at most 2 elements.

Proof.

No element of order > 2 can be central (there is some element inverting it), and if there are at least two elements of order 2, then they can be swapped. □

(Lack of) $\aleph_0$-categoricity

Proposition

If Γ is an infinite inner ultrahomogeneous group, then it is not $\aleph_0$-categorical (i.e. it is not uniformly locally finite).

Proof.

Suppose towards contradiction that Γ is $\aleph_0$-categorical. We will show that it has infinite exponent, contradicting categoricity.
Fix any N. By $\aleph_0$-categoricity and Ramsey's theorem, there is a non-constant indiscernible sequence $g_0, g_1, g_2, \ldots, g_N$ in Γ. By inner ultrahomogeneity, there is some $h \in \Gamma$ such that for $i < N$, $g_i^h = g_{i+1}$. It follows that the order of h is at least $N + 1$. □

Remark

I do not know whether an infinite inner ultrahomogeneous group can have finite exponent.

Quantifier elimination

Proposition

If Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ has disjoint amalgamation, then for any $\bar{a}$ in Γ, we have $\langle \bar{a} \rangle = C(C(\bar{a}))$. (So it is uniformly definable in $\bar{a}$.)

Proof.

$\subseteq$ is trivial. For $\supseteq$, let $g \notin \langle \bar{a} \rangle$ be arbitrary. Then by disjoint amalgamation, there is some $g' \neq g$ such that $\bar{a}g \cong \bar{a}g'$. By inner ultrahomogeneity, there is some h such that $(\bar{a}g)^h = \bar{a}g'$, so $h \in C(\bar{a})$, but $g \notin C(h)$. □

Remark

For a singleton $a = \bar{a}$, it is still true is we assume instead that $\mathtt{Age}(\Gamma)$ is closed under $\times$.

Quantifier elimination

Corollary

If Γ has infinite exponent (+ maybe some technical assumptions), then the condition $x \in \langle y \rangle$ is not quantifier-free definable in Γ.

Proof.

Suppose $\varphi(x, y)$ is a formula which implies that $x \in \langle y \rangle$. Then it implies that x and y commute. Thus, if it is quantifier-free, it is equivalent to the conjunction of $xy = yx$ and a finite boolean combination of formulas of the form $x^n = y^m$. Under suitable assumptions, one can show that these cannot work. □

Corollary

If Γ is as above, inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ has disjoint amalgamation, then Γ does not have quantifier elimination.

Ample generics

Recall the following fact.

Fact (Truss, Ivanov, Kechris-Rosendal)

If M is ultrahomogeneous with age $\mathcal{K}$ and the class $\mathcal{K}_p^n$ of $\mathcal{K}$-structures with n finite partial automorphisms admits a cofinal subclass with AP, then M has a generic n-tuple of automorphisms (i.e. a comeagre diagonal conjugacy class in $\mathtt{Aut}(M)^n$).

Proposition

If Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$ or $*$, the class of (tuples of) inner automorphisms has the AP.

Corollary

Each such Γ has ample generic automorphisms.

Ample generics

Proposition

If Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$ or $*$, the class of (tuples of) automorphisms has the AP.

proof (when Γ is not torsion and age is $\times$-closed).

Fix $G_1, G_2 \leq \Gamma$ and automorphisms $\sigma, \sigma_1, \sigma_2$ of the respective groups, with $\sigma \subseteq \sigma_j$. By inner ultrahomogeneity, there are for $j = 1, 2$ some $g_j \in \Gamma$ such that σ_j is realised by conjugation by g_j. Since Γ is not torsion and the age is $\times$-closed, we may assume without loss of generality that g_1 and g_2 are of infinite order and $\langle g_j \rangle \cap G_j = \{e\}$. But then $\langle G, g_1 \rangle \cong \langle G, g_2 \rangle$, so we can amalgamate $\langle G_1, g_1 \rangle$ and $\langle G_2, g_2 \rangle$, merging g_1 and g_2. □

- The case of tuples follows.
- When Γ is torsion, ensure $\mathtt{ord}(g_1) = \mathtt{ord}(g_2)$.
- When Γ (nontrivial and) with $*$-closed age, choose g_j which doubles the generator of $\mathbb{Z}$ in $G_j * \mathbb{Z}$.

Independence property and the strict order property

Definition

We say that a family of sets is *independent* if every nontrivial finite Boolean combination of them is nonempty.

Definition

Fix a structure M.
We say that a formula $\varphi(x, y)$ has the *IP (independence property)* if for every n, we can find $b_1, \ldots, b_n \in M$ such that the sets $\varphi(M, b_j)$ are independent.
We say that a formula $\varphi(x, y)$ has the *SOP (strict order property)* if it defines a preorder if an infinite chain.

Independence Property

Proposition

Suppose Γ is nontrivial, inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$.
Then the formula $xy = yx$ has the IP (and so it is unstable).

Proof.

Fix n. By hypothesis, we can find a cyclic group C such that $\langle g_1, \ldots, g_n \rangle = C \times C \times \cdots \times C \leq \Gamma$. Any permutation of the generators is a partial automorphism.
If σ is any such permutation and $h \in \Gamma$ is a corresponding witness to inner ultrahomogeneity, then h commutes with g_j if and only if $g_j \notin \mathrm{supp}(\sigma)$. □

(Same argument works if age is $*$-closed, or has some other "symmetric JEP".)

Strict order property

Lemma

Suppose Γ *is inner ultrahomogeneous and* $\mathtt{Age}(\Gamma)$ *is closed under* $\times$*. Then given any* $g \in \Gamma$ *of order* $n < \infty$*, then for every* $m|n$*,* $m > 1$*, there is* $k \in \Gamma$ *commuting with* g^m *but not* g*.*

Proof.

Since $\mathtt{Age}(\Gamma)$ is closed under $\times$, there is some $h \in \Gamma$ of order n, such that $\langle g, h\rangle = \langle g\rangle \times \langle h\rangle$. Let $g' = gh^{n/m}$. Then $(g')^m = g^m$, and by inner ultrahomogeneity, there is some k such that $g^k = g'$, and hence $(g^m)^k = (g')^m = g^m$. □

Lemma

Suppose Γ *is inner ultrahomogeneous and* $\mathrm{Age}(\Gamma)$ *is closed under* $\times$*. Then given any* $g \in \Gamma$ *of order* $n < \infty$*, then for every* $m|n$*,* $m > 1$*, there is* $k \in \Gamma$ *commuting with* g^m *but not* g*.*

Corollary

If, furthermore, Γ *has elements of arbitrarily large finite order, then there are arbitrarily long strict chains of centralisers (so we have the strict order property).*

Proof.

The hypothesis implies that for every n, there is a sequence $p_1, p_2, \ldots, p_n$ of primes and an element $g \in \Gamma$ of order $p_1 \cdots p_n$. Then commutators of $g, g^{p_1}, g^{p_1 p_2}, g^{p_1 p_2 p_3}, \ldots$ are progressively larger. □

- A similar proof works if Γ is neither torsion nor torsion-free (age still $\times$-closed).

Definable ultrahomogeneity

- The notion of inner ultrahomogeneity can be generalised.
- For example, we can assume that there is a (quantifier-free?) definable/interpretable group of automorphisms witnessing ultrahomogeneity.
- Or, we can ask that there be uniformly definable/interpretable family (possibly closed under composition) of automorphisms.
- Under suitable assumptions, we can, at least, use these notions to obtain similar non-tameness conclusions.