Generic embeddings into Fraïssé structures

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Let M be a (classical) Fraïssé structure in a relational language (M is countable and ultrahomogeneous).

Let $A \in \mathrm{Age}(M)$, i.e. A is isomorphic to a finite subset of M.

Then immediately from the definition of ultrahomogeneity, we have that:

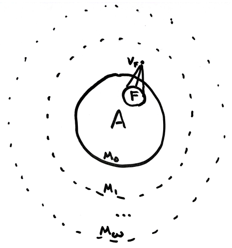
- for any two embeddings $f, f' : A \to M$, there exists $h \in \mathrm{Aut}(M)$ with $hf = f'$;
- for any embedding $f : A \to M$, every automorphism of A extends under f to an automorphism of M.

Naïve question: what happens for infinite A?

Let's look at $M =$ the random graph.

Let A be a countable graph. Then **there exists an embedding $f : A \to M$ such that every element of $\mathrm{Aut}(A)$ extends under f to an element of $\mathrm{Aut}(M)$:**

We construct a countable graph M_ω by induction. Let $M_0 = A$. Assuming M_{i-1} has already been constructed, construct M_i by taking M_{i-1} and, for each finite $F \subseteq_{\text{fin.}} M_{i-1}$, add a new vertex v_F adjacent to exactly F. Then let $M_\omega = \bigcup_{i<\omega} M_i$.



Each $g \in \mathrm{Aut}(A)$ extends to $\mathrm{Aut}(M_\omega)$ (extend shell by shell). It is straightforward to check that M_ω satisfies the "witness property" of the random graph, and therefore $M_\omega \cong M$. We therefore obtain the embedding f desired.

(In fact, here extension commutes with composition: we get a copy of $\mathrm{Aut}(A)$ *inside* $\mathrm{Aut}(M)$ *as a subgroup.)*

This embedding had already been discovered by Henson in 1973!

Such an embedding f, giving a copy of $\mathrm{Aut}(A)$ as a subgroup inside $\mathrm{Aut}(M)$ for all $A \hookrightarrow M$, exists for:

- M having free amalgamation (Bilge & Melleray);
- M having a stationary independence relation (Müller).

The general machinery here is provided by **Katětov functors** (Kubiś, Mašulović).

Okay, so what now?

Question

Was this embedding f "the typical situation" or somehow "weird"? What happens *usually*?

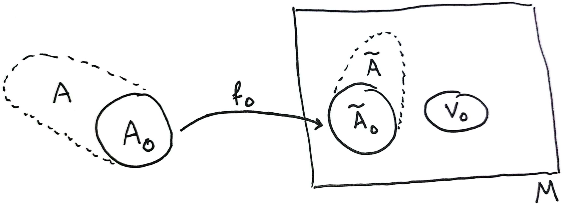
Let M be a Fraïssé structure with strong amalgamation, and let $A \hookrightarrow M$. Let $\mathrm{Emb}(A, M)$ be the set of embeddings of A into M.

We put a natural topology on $\mathrm{Emb}(A, M)$. Given:

- an embedding $f_0 : A_0 \to M$ of a finite subset $A_0 \subseteq A$;
- a finite subset $V_0 \subseteq M$ with $f_0(A_0) \cap V_0 = \varnothing$;

specify a basic open set $[f_0, V_0]$ by:

$$[f_0, V_0] = \{f \in \mathrm{Emb}(A, M) \mid f \text{ extends } f_0 \text{ and avoids } V_0\}.$$



$\mathrm{Emb}(A, M)$ with this topology is a Polish space:

- Enumerate A as $a_0, a_1, \cdots$ and M as $m_0, m_1, \cdots$.
- Given $f, f' \in \mathrm{Emb}(A, M)$.
- Let u be the least index such that $f(a_u), f'(a_u)$ differ, and let v be the least index such that $m_v \in f(A) \triangle f'(A)$.
- Define $d(f, g) = \frac{1}{\min(u,v)}$.

As $\mathrm{Emb}(A, M)$ is a Polish space, the notions of meagre & comeagre behave well.

(Comeagre = common, meagre = uncommon, a subset can't be common AND uncommon.)

We say that *a generic embedding has property P* if

$$\{f \in \mathrm{Emb}(A, M) \mid f \text{ has property } P\}$$

is comeagre.

Let M be a Fraïssé structure with strong amalgamation and let $A \hookrightarrow M$.

Question

Are two embeddings $A \to M$ generically isomorphic? This means: is $\{(f_0, f_1) \in \mathrm{Emb}(A, M)^2 \mid \exists\, h \in \mathrm{Aut}(M) \text{ with } hf_0 = f_1\}$ comeagre?

Question

Let $g \in \mathrm{Aut}(A)^*$. Is g generically extensible? This means: is $\{f \in \mathrm{Emb}(A, M) \mid g \text{ extends under } f \text{ to an element of } \mathrm{Aut}(M)\}$ comeagre?

We have theorems giving clean characterisations for these two questions in terms of the *space of external types*. Today we'll focus on the second question.

As an example, again take $M =$ the random graph and $A \hookrightarrow M$. We will show that **all** $g \in \mathrm{Aut}(A)^*$ **are generically inextensible**.

Fix g. We play the Banach-Mazur game on $\mathrm{Emb}(A, M)$:

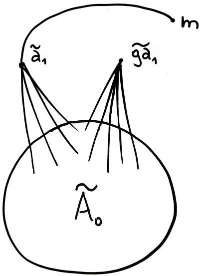
- this is a two-player game;
- the players alternate, and each must give a non-empty open set inside the previous open set given.
- If Player II can always ensure that the intersection of the open sets played consists of embeddings f for which property P holds, then P holds generically.

Here P is the property "g is inextensible under f".

We may assume Player I and Player II play basic open sets $[f_i, V_i]$.

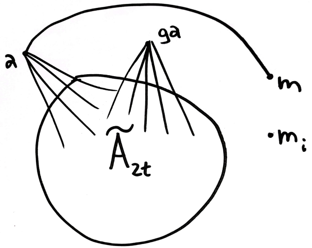
Zeroth turn:

- Player I plays $[f_0, V_0]$. (We write $\widetilde{A_0} = f_0(A_0)$.)
- Player II takes $a_1 \in \operatorname{supp} g$ and embeds a_1, ga_1 in M to produce f_1 (if this hasn't already been done). They then take $m \in M \setminus \widetilde{A_0}$ with $m \sim \widetilde{a_1}, m \not\sim \widetilde{ga_1}$, and place m in $V_1 \supseteq V_0$. This ensures that m cannot be fixed in any automorphic extension of $\widetilde{g}$.
  Player II enumerates $M \setminus \{m\}$ as $m_1, m_2, \cdots$.



ith turn:

- Player I plays $[f_{2t}, V_{2t}]$.
- Player II bans $m \mapsto m_i$ in any automorphic extension of $\widetilde{g}$ by taking $a \in A$ such that $a, ga \notin A_{2t}$, and then embedding a, ga such that $m \sim \widetilde{a}$, $m_i \not\sim \widetilde{ga}$.



At the end of the game, m can't be fixed in any automorphic extension, and m can't be sent to any m_i. So no automorphic extension exists!

For the triangle-free random graph, we also have that all $g \in \mathrm{Aut}(A)^*$ are generically inextensible. The proof is quite a bit harder: when embedding $\widetilde{a} \sim m$, we could accidentally make a triangle.

To fix this, we essentially need to find finite edge-free sets inside A that are not contained in any finite maximal edge-free set.

(Similar idea for the K_n-free random graph, where we also always have g generically inextensible.)

Another example: linear orders. Here $M = \mathbb{Q}$.

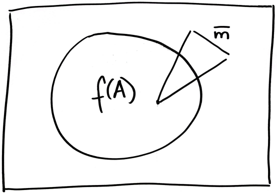
Proposition

Let $M = \mathbb{Q}$, $A \hookrightarrow M$, $g \in \mathrm{Aut}(A)^*$.

- *If $\operatorname{supp} g$ contains a dense interval of A, then g is generically inextensible.*
- *If not, then g is generically extensible.*

General results: consider the space of external (realised, quantifier-free) types.

Let $f \in \mathrm{Emb}(A, M)$ and let $\bar{m} \in (M \setminus f(A))^n$. Then we define $tp_f(\bar{m}/A)$, the *quantifier-free type of* $\bar{m}$ *over* (A, f), to be the set of quantifier-free formulae with parameters in A satisfied by $\bar{m}$ in M.



(Here we consider M as an $\mathcal{L}(A)$ structure by interpreting $a \in A$ as $f(a)$.)

Note that we only consider $\bar{m}$ *external* to $f(A)$.

- We denote the set of external n-types by E_n.
- We denote the set of isolated external n-types by I_n. (We mean isolated in E_n.)
- We refer to elements of $\overline{I_n}$ as *approximately isolated* types.

M a Fraïssé structure with strong amalgamation, $A \hookrightarrow M$.

Theorem

- *If for all n we have $E_n = \overline{I_n}$, then pairs of embeddings $A \to M$ are generically isomorphic.*
- *If not, then pairs of embeddings $A \to M$ are generically non-isomorphic.*

To characterise generic extensibility of $g \in \mathrm{Aut}(A)^*$, we require a new definition.

Definition

Let $p(\bar{x})$ be an external n-type. We say that $p(\bar{x})$ is *losslessly g-split* if there exists a partition $\{1, \cdots, n\} = \beta \sqcup \gamma$ such that:

- $p(x_i : i \in \beta)$ is approximately isolated and $p(x_i : i \in \gamma)$ is g-fixed;
- for any other external n-type $q(\bar{x})$ with $q(x_i : i \in \beta) = p(x_i : i \in \beta)$ and $q(x_i : i \in \gamma) = p(x_i : i \in \gamma)$, we have that $q(\bar{x}) = p(\bar{x})$.

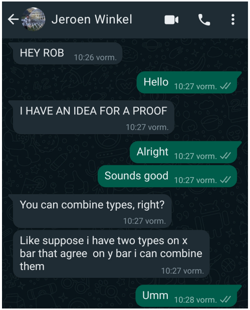
We then have the following theorem:

Theorem

Let M be a Fraïssé structure with strong amalgamation, let $A \hookrightarrow M$ and let $g \in \mathrm{Aut}(A)^*$.

- *If all external types are losslessly g-split, then g is generically extensible.*
- *Assume M has free amalgamation. If there exists an external type which is not losslessly g-split, then g is generically inextensible.*

In the second statement, we can probably weaken free amalgamation to just requiring some kind of canonical amalgamation (a SWIR?): ongoing work.



I would like to thank my collaborators Alessandro Codenotti, Aristotelis Panagiotopoulos and Jeroen Winkel.

(Jeroen has continued to contribute to the project even though he has left academia, mostly via Whatsapp messages.)

Thanks very much for listening,
and have a good rest of the conference!