On extending Cantor subsystems on dendrites

Jakub Tomaszewski

26.10.2023

joint work with Dominik Kwietniak and Piotr Oprocha

Backstory

Entropy paradox

In the following dynamics complexity hierachy

$$\text{transitivity} \leq \text{mixing} \leq \text{exactness}$$

exact maps of an interval, a circle[1] and topological graphs[2] can attain lower entropy than mixing maps.

Theorem (H-K-O)

Let T_k be a full binary tree of height k. For each $\epsilon > 0$ there exists an exact map $f_\epsilon \colon T_k \to T_k$ with $h_{\text{top}}(f_\epsilon) \leq \frac{\log 3}{2^n} + \epsilon$.

[1] Grzegorz Harańczyk and Dominik Kwietniak. "When lower entropy implies stronger Devaney chaos". In: *Proc. Amer. Math. Soc.* 137.6 (2009), pp. 2063–2073.

[2] Grzegorz Harańczyk, Dominik Kwietniak, and Piotr Oprocha. "Topological structure and entropy of mixing graph maps". In: *Ergodic Theory Dynam. Systems* 34.5 (2014), pp. 1587–1614.

Definition

Gehman dendrite $\mathcal{G}$ is the unique dendrite with each branching point of order 3 and the set of endpoints homeomorphic to the Cantor set.

Theorem (K-O-T)

Let $\epsilon > 0$. There exists a mixing map $F_\epsilon \colon \mathcal{G} \to \mathcal{G}$ with $h_{\mathrm{top}}(F_\epsilon) \leq \epsilon$.

Corollary

The entropy paradox does not hold on dendrites.

Observation

Cantor systems naturally live in $\mathcal{G}$.

Question 1

Can we extend an endpoint subsystem of choice onto the whole dendrite?

Question 2

For which Cantor systems can we do that?

Cantor subsystems

Graph covers[3]

Definition

We say that a graph homomorphism $\phi\colon V_1 \to V_2$ between two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ is a (graph) cover if

1. ϕ is vertex-surjective: $\phi(V_1) = V_2$,
2. ϕ is $+$ directional: $(x, y), (x, z) \in E_1 \implies \phi(y) = \phi(z)$.

Definition

For a sequence of graph covers $(G_i, \phi_i)_{i=0}^{\infty}$ define

$\mathrm{G} = G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} G_3 \xleftarrow{\phi_3} \ldots$,

$V_{\mathrm{G}} = \{(x_0, x_1, x_2, \ldots) \in \prod_{i=0}^{\infty} V_i : \ x_i = \phi_i(x_{i+1}) \text{ for } i \in \mathbb{N}\}$ and

$E_{\mathrm{G}} = \{(x, y) \in V_{\mathrm{G}} \times V_{\mathrm{G}} : \ (x_i, y_i) \in E_i \text{ for } i \in \mathbb{N}\}$.

[3] Takashi Shimomura. "Special homeomorphisms and approximation for Cantor systems". In: *Topology Appl.* 161 (2014), pp. 178–195.

Theorem (S)

The space V_{G} is a compact metrizable 0-dimensional space and E_{G} is a continuous map of V_{G}.

Theorem (S)

Let $(\mathcal{C}, f)$ be a surjective Cantor system. Then there exists a refining sequence of decompositions $(\mathcal{U}_i)$ such that

1. the sequence $(\mathcal{U}_1, f^{\mathcal{U}_1}) \xleftarrow{\phi_1} (\mathcal{U}_2, f^{\mathcal{U}_2}) \xleftarrow{\phi_1} \ldots$, where $U \sim_{f^{\mathcal{U}}} V \iff f(U) \cap V \neq \emptyset$, of natural graph homomorphisms is a sequence of covers,
2. the inverse limit of this sequence of covers is conjugate to (X, f).

Theorem (S+T)

Let $(\mathcal{C}, f)$ be a surjective Cantor system and $\mathcal{U}_0$ a decomposition of $\mathcal{C}$. Then the following sequence

$$\mathrm{G} = (\mathcal{C}, (\mathcal{C}, \mathcal{C})) \xleftarrow{\phi_0} (\mathcal{U}_1, f^{\mathcal{U}_1}) \xleftarrow{\phi_1} (\mathcal{U}_2, f^{\mathcal{U}_2}) \xleftarrow{\phi_2} \dots,$$

where each homomorphism ϕ_i is induced by the identity map on $\mathcal{C}$ and each decomposition $\mathcal{U}_{i+1}$ is a refinement of

$$f^{-1}(\mathcal{U}_i) \vee \mathcal{U}_i = \{U \cap V : \ U \in f^{-1}(\mathcal{U}_i), \ V \in \mathcal{U}_i\}$$

with $mesh(\mathcal{U}_i) \to 0$, approximates the dynamics of $(\mathcal{C}, f)$.

Shimomura tower

Crucial features

Graph homomorphisms

We choose maps induced by $id\colon \mathcal{C} \to \mathcal{C}$.

Decompositions

We define $(\mathcal{U}_i)_{i=1}^{\infty}$ such that

1. $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i \vee f^{-1}(\mathcal{U}_i)$ [chosen graph homomorphisms are covers],
2. $mesh(\mathcal{U}_i) \to 0$ [the system $(V_{\mathrm{G}}, E_{\mathrm{G}})$ approximates $(\mathcal{C}, f)$],
3. each $U_i \in \mathcal{U}_i$ is split into at least four different sets from $\mathcal{U}_{i+1}$ [secret tool for later].

Algorithm

1. Start off on a level where you finished,
2. Find such depth that each disintegration of previous set fits in the same subtree,
3. Assign a branching point for each set.

Theorem

The dendrite $\mathcal{S}$ obtained by taking all infinite paths created by the algorithm is homeomorphic to $\mathcal{G}$.

Algorithm

1. Map branching points into dynamical ancestors,
2. Stretch edges over between assigned endpoint images.

Theorem

The endpoint subsystem of $(\mathcal{S}, F)$ *is conjugate to* $(\mathcal{C}, f)$.

Algorithm

1. Stretch nonsignificant edges over maximal tree between starting level and the image root,
2. Stretch significant edges over all the related paths, exceeding the starting level.

Theorem

The modified map F_{mix} is mixing.

Interesting dynamics

Surjective limit map[4]

Theorem (N)

Let X, Y be compact metric spaces and $F_n\colon X \to 2^Y$ be a countable sequence of upper semicontinuous maps with

1. $F^{n+1}(x) \subset F_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$,
2. $\lim_{n\to\infty} diam(F_n(x)) = 0$ for all $x \in X$,
3. $F_n(X) = Y$ for all $n \in \mathbb{N}$.

Then the map $f\colon X \to Y$ defined by $f(x) = \bigcap_{n\in\mathbb{N}} F_n(x)$ for every $x \in X$ is continuous and onto.

[4] Sam B. Nadler Jr. *Continuum theory*. Vol. 158. Monographs and Textbooks in Pure and Applied Mathematics. An introduction. Marcel Dekker, Inc., New York, 1992.

Algorithm

Recurrently map subintervals of the first edges into subdendrites, stretching them over level by level.

Theorem

The modified map F_{ex} is exact.

Wrap up

What we showed

Let $(\mathcal{C}, f)$ be a surjective Cantor system. Then

1. There exists a $(\mathcal{U}_i)$ cover setup in which the Shimomura tower can be constructed in a natural way,
2. This tower approximates well the action on the Cantor system,
3. Following the above, there exists a way to define the dynamics on $\mathcal{G}$ in such a way that the endpoint subsystem is conjugate to $(\mathcal{C}, f)$,
4. Without the change of the above property, the defined map can be made mixing,
5. Or even exact.

Thank you!