A sequence of coins and a Rrandom Graph

Agnieszka Widz

Institute of Mathematics
Łódź University of Technology

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joint work with Leonardo N. Coregliano and Jarosław Swaczyna

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Basic informations

The Rado Graph, sometimes also known as the (countable) Random Graph, can be generated almost surely by putting an edge between any pair of vertices with some fixed probability $p \in (0, 1)$, independently of other pairs.

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Definition

We say that a graph (V, E) satisfies the property $\star$ if

$$\text{For all finite disjoint } A, B \subseteq V \text{ there is a vertex } v \in V \text{ such that } v \text{ is connected to all elements of } A \text{ and to no element of } B. \qquad (\star)$$

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Two graphs with property $\star$ are isomorfic.

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Example 1: $(p_n) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2} \dots)$
Sequence of pairs vertices:
$(\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_6, v_{15}\}, \{v_1, v_4\} \dots)$

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Example 2: $(p_n) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2} \dots)$
Sequence of pairs vertices:
$(\{v_7, v_9\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_6, v_{15}\} \dots)$

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For which sequences of coins there exist such permutation of pairs of vertices that we get the Random Graph?

Tools

Lemma 1

Let $(p_n)_{n\in\mathbb{N}}$ be a non-increasing sequence of elements of interval $[0,1]$ and let $k \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} p_n = 0$ and $\sum_{n=0}^{\infty} p_n^k = \infty$. Then

$$\sum_{n=0}^{\infty} \prod_{i=0}^{k-1} p_{nk+i} = \infty.$$

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Lemma 2

Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of elements in the interval $[0,1]$. Suppose that $\lim_{n\to\infty} p_n = 0$ and $\sum_{n=0}^{\infty} p_n^k = \infty$ for all $k\in\mathbb{N}$.
Then $(p_n)_{n\in\mathbb{N}}$ may be split into infinitely many subsequences $((p_{s_\ell^{m,k}})_\ell)_{m,k}$ such that

$$\sum_{\ell=0}^{\infty} p^k_{s_\ell^{m,k}} = \infty$$

for every $m,k\in\mathbb{N}$.

Tools

Lemma 3

If for each natural number k the sums $\sum_{n=0}^{\infty} p_n^k$ and $\sum_{n=0}^{\infty}(1-p_n)^k$ are infinite, then there exists an injection $f\colon \mathfrak{X} \to \mathbb{N}$ such that

$$\forall k, m \in \mathbb{N}, \sum_{n=0}^{\infty} \prod_{i=0}^{k-1} p_{f(k,m,n,i)} \prod_{i=k}^{2k-1} (1 - p_{f(k,m,n,i)}) = \infty,$$

and $\mathbb{N} \setminus \operatorname{rng}(f)$ is infinite, where

$$\mathfrak{X} := \{(k, m, n, i) \in \mathbb{N}^4 : i \leq 2k - 1\}.$$

Solution to the puzzle

Main Theorem

The following are equivalent for a sequence $(p_n)_{n\in\mathbb{N}}$ of numbers from interval $[0,1]$.

1. There exists a bijective assignment $f\colon [\mathbb{N}]^2 \to \mathbb{N}$ such that by letting each $\{v,w\} \in [\mathbb{N}]^2$ be an edge with probability $p_{f(\{v,w\})}$, independently from other pairs, the resulting graph is the Random Graph with probability 1.
2. Item (1) holds but the conclusion holds with positive probability instead of probability 1.
3. For every $k \in \mathbb{N}$, the sums $\sum_{n=0}^{\infty} p_n^k$ and $\sum_{n=0}^{\infty}(1-p_n)^k$ are infinite.

THANK YOU!