Furthermore, we can prove (see [9], p. 16) that for $a_{\alpha_t\alpha}(t, x, \eta) - b_x(t, x, \eta) \underset{(<)}{>} 0$ and $h \underset{(<)}{>}$ the function

$$\tilde{x}_0(t) = \begin{cases} x_0(t) & \text{in } G \setminus U(t_0, h), \\ \psi_{1(2)}(t, h, \eta) & \text{in } U(t_0, h) \end{cases}$$

is an element of $\mathfrak{A}_2$ satisfying the relation $J_2(\tilde{x}_0) < J_2(x_0)$. But this is a contradiction of the minimal property of x_0 with respect to problem (3). This means that for $x_0(t)$ there are no points $t_0 \in \mathfrak{N}_0$ at which $\eta = x_{0t}(t_0)$ realizes case II of Hypothesis II.

With the result obtained before we know under Hypothesis II that case I is valid for every t_0 with the corresponding derivative $\eta = x_{0t}(t_0)$. Then the constructed neighbourhoods $U(t_0, h)$, here considered for all $t_0 \in \mathfrak{N}_0$, define a covering of $\mathfrak{N}_0$. By using Vitali's covering theorem we can choose a countable set of disjunct neighbourhoods $U_j = U(t_j, h_j)$ in the sense above, such that mes $[\mathfrak{N}_0 - \bigcup U_j] = 0$. If we replace $x_0(t)$ in U_j by $\psi_1(t, h_j, \eta_j)$ with $\eta_j = x_{0t}(t_j)$, then we obtain, analogously to [9], a new solution $\tilde{x}_0$ of problem (3) with $\tilde{x}_{0t}(t) \in \mathfrak{M}(t, \tilde{x}_0(t))$ almost everywhere on G. From conclusion 2 it follows that the function $\tilde{x}_0$ is a solution of problem (2) as well, and thus the proof of our theorem is finished.

Remark. It is possible to give examples in which Hypothesis I is fulfilled but no solutions of (2) exist because Hypothesis II is not valid (see [9], p. 8).

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BANACH CENTER PUBLICATIONS
VOLUME 1

APPROXIMATION METHODS FOR NONLINEAR PROBLEMS WITH CONSTRAINTS IN FORM OF VARIATIONAL INEQUALITIES

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We consider here a number of nonlinear problems, which can all be written in the form of mixed variational inequalities. We assume also the constraints on these problems to be expressed in the form of mixed variational inequalities.

In other words, we consider mixed variational inequalities on sets of solutions of other variational inequalities.

In this paper we confine ourselves to the case where the operators and functionals figuring in the variational inequalities are monotone and convex, respectively. Generalizations into various directions are possible.

Part 1 contains an existence theorem for solutions of variational inequalities; we shall show that many problems can be considered as such inequalities. In Part 2 we give some qualitative statements concerning variational inequalities on solution sets of other variational inequalities. As an example an optimal control problem for a linear operator equation is mentioned. Part 3 concerns certain projection methods for the approximate solution of the general problem stated in Part 2. In Part 4 iterative projection methods for the same problem are considered. Finally, Part 5 contains some remarks on generalizations of the theory.

1. Variational inequalities

Let B be a real reflexive Banach space with dual space B^*. By (w, v) we denote the value of the functional $w \in B^*$ at the element $v \in B$. Let $C \subset B$ be a non-empty set, $S \in (B \to B^*)$ an operator, z an element of B^* and $h(x)$ a proper functional, i.e., $h \in (B \to R_1)$, $h(x) > -\infty$, $h(x) \not\equiv +\infty$.

We consider the following problem. Find $u \in C$ satisfying

(1) $$(z - Su, u - v) \geqslant h(u) - h(v) \quad \text{for all } v \in C.$$

Relation (1) is called a *(mixed) variational inequality*. Such an inequality was introduced first by Lescarret in 1965 for the case of a Hilbert space without constraints ($C = B = H$).

THEOREM 1 (Existence). *Let S be monotone and hemicontinuous, h convex and lower semicontinuous, C convex and closed, and let one of the following conditions,* (a) *or* (b), *be fulfilled*:

(a) *C is bounded;*

(b) *there exists a $v_0 \in C$ such that $h(v_0) < +\infty$ and*

$$\frac{(Sv, v-v_0)+h(v)}{\|v\|} \to +\infty \quad as \quad \|v\| \to +\infty,\ v \in C.$$

Then, for every $z \in B^*$, (1) *has at least one solution. The set K_z of all solutions of* (1) *is convex, closed and bounded. If, additionally, S is strictly monotone or h is strictly convex, then* (1) *has a unique solution.*

Proof. See [4], [5], [3], [20], [18].

In the case $B = H$ (Hilbert space) the variational inequality (1) with the identity mapping S ($S = I$) has a unique solution $u = P(C, h)z \in C$ for every $z \in H$. In H problem (1) and the fixed point problem

$$u = P(C, th)\,\big(I-t(S-z)\big)u, \qquad t > 0 \text{ fixed}, \tag{1.1}$$

are equivalent. The operator $P(C, h) \in (H \to C)$ is Lipschitz-continuous with the Lipschitz constant $L\big(P(C, h)\big) = 1$. Especially, $P(C, 0)$ is the projection operator onto the convex set C. If B is a strictly convex space with a strictly convex dual B^*, then also inequality (1) can be transformed into an equivalent fixed point problem, similar to (1.1).

We now consider various nonlinear problems, which lead to special cases of (1).

(a) *Operator equations*

$$Su = z. \tag{1.2}$$

Examples for (1.2) are:

(a1) *classical boundary value problems for nonlinear elliptic partial differential equations in divergent form*, e.g., *the Dirichlet problem*

$$(\bar{S}u)(r) \equiv \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha S_\alpha\big(r, s(u)(r)\big) = z(r) \tag{1.3}$$

with $r \in G \subset R_n$, $s = \{s_\alpha\colon |\alpha| \leqslant m\}$, $s(u)(r) = \{D^\alpha u(r)\colon |\alpha| \leqslant m\}$ and the boundary conditions

$$D^\beta u(r)|_{\partial G} = 0, \qquad |\beta| \leqslant m-1$$

and

(a2) *mixed problems for pseudo-parabolic partial differential equations*:

$$S_1(t)\frac{d}{dt}S_2(t)u(t) = S_3(t)u(t), \qquad S_2(0)u(0) = u_0 \in H_0, \tag{1.4}$$

$$u(t)\colon [0, T] \to H_0, \qquad B = L_2(e^{-2at}; 0, T; H_0) \qquad ([19]).$$

(b) *Variational inequalities of the form*

$$(z-Su, u-v) \geqslant 0, \qquad v \in C. \tag{1.5}$$

Inequalities of this type arise, for instance, from the problems (1.3) and (1.4) with additional constraints on the boundary ∂G (nonclassical boundary value problems, such as the unilateral boundary value problem and the thin obstacle problem), on the region G (elastic plastic torsion) or on parts of G (obstacle problems). Moreover, (1.5) may be a necessary and in some cases also a sufficient condition for a minimum problem (cf. (c)).

(c) *The minimum problem*

$$h(u) = \inf_{v \in C} h(v), \tag{1.6}$$

for instance,

(c1) *nonclassical variational problems*, i.e., (1.6) where

$$h(v) = \int_G H\big(r, s(v)(r)\big)\,dr, \qquad B = \mathring{W}^m_{p_\alpha}(G). \tag{1.7}$$

If $\partial H/\partial S_\alpha = S_\alpha$ and $C = B$, then (1.3) represents the Euler–Lagrange equation for (1.6)–(1.7).

If $C \neq B$ and the gradient h' of h exists, then we have as a necessary condition

$$\big(h'(u), v-u\big) \geqslant 0, \qquad v \in C,$$

which corresponds to (1.5) and is also sufficient for convex functionals h.

(c2) *Problems of optimal control with operator equations* (for example (1.3) or (1.4)), where $h(v)$ is given by

$$h(v) = J(x[v], v) = J_1(x[v])+J_2(v),$$

$x[v]$ being the solution of

$$S(x, v) = 0$$

for fixed v.

If $x[v]$ is a single-valued mapping, $S(\cdot\,,\cdot)$ is a linear mapping and J_1, J_2 are convex functionals, then h is convex. If $x[\cdot]$, J_1 and J_2 are continuous, the functional h is also continuous. If, on the contrary, $S(\cdot\,,\cdot)$ is nonlinear, then, in general, h is not convex. In this case let us suppose that

J_1 is lower semicontinuous, J_2 is weakly lower semicontinuous, and $x[v]$ transforms every weakly convergent sequence $\{v_n\}$ into a strongly convergent sequence $x[v_n]$ (sufficient conditions for these suppositions to be fulfilled are known). Then $h(v)$ is weakly lower semicontinuous, and we can apply the Weierstrass theorem.

Remark. Approximation methods for the solution of (1.5) and (1.6) are given, for instance in [7]–[9], [12], [17], [18], [22], [23]. They are applicable to (c2) (see [11], [15]). If the gradient

$$h'(v) = J'_v(x[v], v) - S'_v(x[v], v)^*[S'_x(x[v], v)^{-1}]^* J'_x(x[v], v)$$

is used in an iteration method, then at every step of the method we have to solve problems in infinite-dimensional spaces. This can be done by approximation methods, solving operator equations which are special cases of (1.2) [9].

We mention here another possibility: the combination of approximation methods for minimum problems (1.6) with approximation methods for the process relations (i.e., for the corresponding operator equations). This combination is an application to problems of optimal control of the methods proposed below (Parts 2–3).

2. Variational inequalities on solution sets of variational inequalities

We consider

$$(g-Tx, x-y) \geqslant f(x)-f(y), \qquad y \in K_z, \tag{2}$$

where $T \in (B \to B^*)$ is monotone and hemicontinuous, $g \in B^*$, $f \in (B \to R_1)$ convex, lower semicontinuous and proper, and $K_z = \{w \in C\colon\ w \text{ is a solution of } (1)\}$.

Inequality (2) characterizes certain solutions of (1). On the other hand, (1)–(2) is a mixed variational inequality (2) with general constraints (1) containing the problems cited above and further special cases. As an example, we mention the problem of optimal control (OS):

(OS) *Let* $M \subset H_0 \times B_0$, $F \in (H_0 \times B_0 \to R_1)$ *and* $S \in (H_0 \times B_0 \to H_0)$. *Find* $(x, u) \in M$ *such that*

$$F(x, u) = \inf_{(y,v)\in M} F(y, v),$$
$$S(y, v) = 0.$$

(OS) is equivalent to the following minimum problem on the solution set of another minimum problem (MP):

$$F(x, u) = \inf_{(y,v)\in K} F(y, v),$$

$$\text{(MP)} \quad K = \{(y, v) \in M\colon\ H(y, v) = \inf_{(\bar{z},\bar{w})\in M} H(\bar{z}, \bar{w}) \text{ where } H(\bar{z}, \bar{w}) = \tfrac{1}{2}\|S(\bar{z}, \bar{w})\|^2_{H_0}\}.$$

For a linear and continuous operator S the functional H is convex and lower semicontinuous and therefore weakly lower semicontinuous. H has the latter property also in the case of a nonlinear operator S, when, additionally, S is weakly continuous. In this case we are dealing with minimum problems on the solution sets of minimum problems for weakly lower semicontinuous functionals. We do not consider here approximation methods for such problems. They can be formulated in the same form as the methods given in Part 3, and they converge under similar assumptions [14].

3. Projection approximation methods

For every natural number n let the mappings $S_n, T_n \in (B \to B^*)$ be monotone and hemicontinuous, the functionals $h_n, f_n \in (B \to R_1)$ convex and lower semicontinuous, $\varepsilon_n > 0$ ($\varepsilon_n \to 0$ for $n \to \infty$). We introduce the notations $A_n = S_n + \varepsilon_n T_n$, $j_n = H_n + \varepsilon_n f_n$ and $a_n = z_n + \varepsilon_n g_n$ where $z_n, g_z \in B^*$.

Let (B_n) be a monotone sequence of subspaces $B_n \subseteq B$, $C_n = C \cap B_n$ and $C_1 \neq \emptyset$.

We consider

$$(a_n - A_n u, u-v) \geqslant j_n(u) - j_n(v), \qquad v \in C_n. \tag{1n}$$

(1n) is, in general, a projection regularization method for (1) and at the same time a projection penalty method for (2) with constraints (1). In particular, for $B_n = B$, (1n) contains methods of penalty operators and penalty functionals and methods of elliptic and Tychonoff regularization. On the other hand, (1n) represents combinations of these methods with projection methods and, in the case of $C = K_Z$, projection methods for (2), such as the Ritz- and Galerkin-methods (see, e.g., [9], [22]). By using abstract difference schemes ([1], [2], [23], [24]), it is possible to extend (1n) to more general projection methods, but we shall not consider this subject here (see [16]).

We now introduce some concepts needed to prove the convergence of the solutions of (1n) to the solutions of (1)–(2).

Definition 1. The sequence $\{A_n\}$ is said to *converge to* S *a-continuously* (*approximation-continuously*) *with respect to* $\{B_n\}$ if we have $w_n \to w$, $w_n \in B_n \Rightarrow A_n w_n \to Sw$. The sequence $\{j_n\}$ is said to *converge weakly lower semi-a-continuously* to h (with respect to $\{B_n\}$) if, for $w_n \to w$ weakly, $w_n \in B_n$,

$$\varliminf_{n\to\infty} j_n(w_n) \geqslant h(w)$$

is satisfied (cf. [12], [14]) for sufficient conditions for these properties to hold).

Definition 2. C is said to be (C_n)-*approximable* if for every $w \in C$ there exists a sequence $\{w_n\}$ such that $w_n \in C_n$ and $\|w_n - w\| \to 0$ (for sufficient conditions see [9]).

Definition 3. $u \in C$ is said to be $(v_n, \varepsilon_n, C_n, z_n - S_n, h_n)$-*approximable with respect to* $\{u_n\}$ ($u_n \in C_n$) if there exists a sequence $\{v_n\}$ satisfying

(a) $v_n \in C_n$,

(b) $v_n \to u$ and

(c) $\displaystyle \varlimsup_{n\to\infty} \left\{\frac{1}{\varepsilon_n}\,[(z_n - S_n v_n, u_n - v_n) + h_n(v_n) - h_n(u_n)]\right\} \leqslant 0.$

(Sufficient conditions are contained, for instance, in [12], [14]).

We remark that another sufficient condition for (c) to hold is the following: *v_n is a solution of*

$$(z_n - S_n v, v - \bar{v}) \geqslant h_n(v) - h_n(\bar{v}), \qquad \bar{v} \in C_n.$$

This remark applies to (OS) and (MP) if one takes

$$h_n(\bar{z}, \bar{w}) \equiv H_n(\bar{z}, \bar{w}) = \tfrac{1}{2}\|P_n S(\bar{z}, \bar{w})\|^2_{H_0}$$

(in this case (1n) combines a projection method for the minimum problem with a projection method for the process equation) and if we have

$$S(\bar{z}, \bar{w}) = S_1(\bar{z}) + S_2(\bar{w})$$

where S_1 and S_2 are linear and continuous operators, S_1 is positive definite, P_n denotes the orthogonal projection operator onto $H_{0_n} \subset H_0$, $M = H_0 \times M^2$, $M^2 \subset B_0$, $M_i^2 = M^2 \cap B_{0_i}$ and $M_i^2 \neq \emptyset$ hold, if there exists a sequence $\{\overline{w}_i\}$ satisfying $\overline{w}_i \in M_i^2$, $\overline{w}_i \to u \in B_0$ (where u is the second component of a solution (x, u) of (OS)) and, finally, if $v_i = (\overline{z}_i, \overline{w}_i)$ where $\overline{z}_i$ denotes the unique solution of the operator equation in H_i

$$P_i S(z, w) = 0.$$

In the case described here (1n) has the form

$$H_n(x_n, u_n) + \varepsilon_n F(x_n, u_n) = \inf_{(\overline{y}, \overline{v}) \in H_{0_i} \times M_i^2} [H_n(\overline{y}, \overline{v}) + \varepsilon_n F(\overline{y}, \overline{v})]. \tag{3.1}$$

For weakly continuous nonlinear operators S one can proceed in an analogous way.

THEOREM 2. *Let the following conditions be satisfied*:

1. $\{A_n\}$ *converges to* S *a-continuously with respect to* $\{B_n\}$; $\{j_n\}$ *converges to* h *weakly lower semi-a-continuously and upper semi-a-continuously with respect to* $\{B_n\}$, $||z_n - z|| \to 0$ *and* $||g_n - g|| \to 0$.

2. (1n) *has at least one solution* u_n *for all* n, (1)–(2) *has a solution* u *which is* $(v_n, \varepsilon_n, C_n, z_n - S_n, h_n)$*-approximable with respect to* $\{u_n\}$.

3. C *is* (C_n)*-approximable if (for all* n) $z_n \neq S_n$.

4. $f_n = f$ *for all* n, *and* f *is uniformly convex or*

4′. $T_n = T$ *for all* n, *and* T *is uniformly monotone.*

Then the sequence $\{u_n\}$ *converges strongly (in* B) *to* u.

Proof. See [14]. Special cases of Theorem 2 are contained in [7], [9], [12]. In [14] Theorem 2 is formulated in a more general way.

For strongly convex functionals H Theorem 2 implies results on the convergence of the solution of (3.1) to the unique solution of (OS).

As the example of the optimal control problem (OS) shows, Theorem 2 can be applied directly (cf. [9], [11], [19]). If this is not possible, then one may take $B_n = B$ and apply the usual projection method to (1n).

4. Iterated approximation methods

We shall now combine method (1n) with an iteration process. This leads to several types of iterated projection approximation methods.

Let M be a metric space, $\{A_i\}$ a sequence of mappings, $A_i \in (M \to M)$, $\{x_i\}$ a sequence defined by

$$x_i = A_i x_{i-1}, \quad x_0 \in M, \tag{4.1}$$

where $T_i \equiv A_i \ldots A_1$ is lipschitzian with Lipschitz constant $L(T_i) \to 0$ as $i \to \infty$.

Let $u_0 \in M$ be an element satisfying (in the sense of the metrics of M) $T_i u_0 \to u_0$ for $i \to \infty$. Then $\{x_i\}$ converges to u_0 in M (cf. [10], [12]). In [9], [12], [14], and [17], [18], several special variants of (4.1) are considered, statements concerning the speed of convergence are derived and generalizations to locally lipschitzian mappings A_i are given. For applications see [9], [10], [12].

If one takes $B = H$ in the conditions of Part 3, then the sequence of fixed point problems

$$u_n = P(C_n, t_n j_n) P_n \big(I - t_n(A_n - a_n)\big) u_n$$

is equivalent to the method (1n).

For instance, let A_n be strongly monotone and lipschitzian. Then, under the assumptions of Theorem 2, the iterated projection method

$$x_i = [P(C_i, t_i j_i) P_i (I - t_i(A_i - a_i))]^{n_i}, \quad x_{i-1}, x_0 \in M, \tag{4.2}$$

converges in H to the same solution u of (1)–(2) as the method (1n). Here n_i is the smallest integer not less than $e_0 / (1 - L(W_i))$, where e_0 is a fixed positive number and $L(W_i)$ is the Lipschitz constant of $W_i = I - t_i(A_i - a_i)$, and t_i is the iteration parameter satisfying

$$0 < t_i < \frac{2c(A_i)}{L(A_i)^2} \qquad (c(A_i) \text{ is the monotonicity constant of } A_i).$$

The convergence result cited here is proved in [14]. For special cases see [7], [9], [12].

Method (4.2) contains such procedures as the well-known projected-gradient method, methods of iterated penalisation and iterated regularization and their combinations with projection methods, for example projection-iteration methods.

(4.2) can also be applied to (OS) if one proceeds from (3.1) to equivalent variational inequalities (using here the differentiability of the functionals occurring in (3.1)). In this way a projected penalty method for (OS) is obtained which has to be iterated.

Similarly to (4.2), the above-mentioned generalizations of (1n) by abstract difference schemes can be iterated.

In Hilbert spaces, the iteration of methods includes an effect of linearization, very useful for numerical realization. Such an iteration can also be carried out for mappings which are not locally lipschitzian [8], for pseudomonotone mappings, and in some cases (projection-iteration methods) even in Banach spaces [12], [14].

5. Concluding remarks

In formulating the optimal control problem (OS) we used mappings from the whole space $H_0 \times B_0$ into H_0. This makes it possible to cover such problems as, for instance, optimal control of classical boundary value problems for elliptic partial differential equations and for pseudo-parabolic partial differential equations.

However, the scheme given in Parts 2 and 3 may be extended to problems where the mappings are defined only on a dense set. Then it can also be applied to control problems for parabolic partial differential equations ([13]). In [15] some optimal control problems for variational inequalities representing nonclassical (for instance, unilateral) boundary value problems for partial differential equations are considered.

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BANACH CENTER PUBLICATIONS
VOLUME 1

DEUX REMARQUES SUR LA COMMANDE BANG BANG DES SYSTÈMES SEMI LINÉAIRES

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Le but de cette note est de faire quelques remarques sur la contrôlabilité Bang Bang des systèmes semi linéaires de la forme:

$$\frac{dx}{dt} = f^0(x) + \sum_{i=1}^{i=p} u_i f^i(x), \qquad x \in \boldsymbol{R}^n. \tag{1}$$

Nous appellerons *commande admissible* une application constante par morceaux d'un intervalle de R dans le cube unité de $\boldsymbol{R}^p$:

$$U = \{u_1, u_2, \ldots, u_p;\ |u_i| \leqslant 1\},$$

et une *commande Bang Bang* sera une fonction constante par morceaux prenant ses valeurs dans les sommets du cube unité, soit:

$$U_{\text{B.B.}} = \{(u_1, u_2, \ldots, u_p) \in U;\ u_i = +1 \text{ ou } -1\};$$

le fait de choisir pour commandes admissibles des commandes constantes par morceaux au lieu de commandes différentiables par morceaux comme il est plus fréquent de le faire, apporte une petite simplification à ce qui va suivre mais n'est pas une restriction essentielle. Si $t \to \mathscr{U}(t)$ est une commande admissible, on note sa réponse: $t \to x(t, x_0, \mathscr{U})$, c'est, par définition, la solution du problème de Cauchy:

$$\frac{dx}{dt} = f^0(x) + \sum_{i=1}^{i=p} u_i(t) f^i(x),$$

$$x(0) = x_0,$$

avec:

$$t \to \mathscr{U}(t) = \big(u_1(t), u_2(t), \ldots, u_p(t)\big).$$

L'ensemble des états accessibles à l'instant T noté $A(T, x_0, U)$ est défini par:

$$A(T, x_0, U) = \big\{x(T, x_0, \mathscr{U});\ \mathscr{U} \in \{\text{commandes admissibles}\}\big\}.$$

L'ensemble des états accessibles $A(x_0, U)$ est la réunion pour $T \geqslant 0$ des $A(T, x_0, U)$. On définit de la même manière les états accessibles par des commandes Bang Bang,