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# NECESSARY OPTIMALITY CONDITIONS FOR A MODEL OF OPTIMAL CONTROL PROCESSES

LEONHARD BITTNER

*Sektion Mathematik, Ernst-Moritz-Arndt Universität, Greifswald, DDR*

## 1. The problem

If nonlinear multidimensional integral equations serve as the mathematical tool, describing a process, one is involved in a lot of irksome calculations, which suggest the adoption of an abstract point of view, introduction of a model of the process and investigation of the model instead of the actual process. This paper presents such a model. In order to formulate it, let us assume that $X$ denotes a (real) Banach space, $M$ a certain subset of $X$, $W$ an arbitrary set, $T$ a mapping of $X\times W$ into $X$, and $f$ a mapping of $X\times W$ into the set $\boldsymbol{R}$ of real numbers.

The problem under consideration is:

*Minimize* $f(x, w)$ *subject to* $x = T(x, w)$, $w \in W$, $x \in M$.

We interpret $x$ as a state, $w$ as a control, $W$ as the set of feasible controls, $M$ as a state constraint set, the fixed point equation $x = T(x, w)$ as the equation of a process, and $f(x, w)$ as a performance index.

## 2. Assumptions

In order to grasp both the process equation and the performance index simultaneously, it is convenient to introduce an auxiliary real variable $\xi$, to pose

(1) $$y = (x, \xi), \quad Y = X\times \boldsymbol{R}, \quad S(y, w) = \big(T(x, w), f(x, w)\big)$$

and to study the fixed point equation $y = S(y, w)$, which is equivalent to the two relations

(2) $$x = T(x, w), \quad \xi = f(x, w).$$

We consider $Y$ a (complete) normed space under the norm $\|y\| = \|x\|+|\xi|$ and a pseudonormed space (a $B_k$-space) under the pseudonorm $|y| = (\|x\|, |\xi|)^{\mathrm{T}} \in \boldsymbol{R}^2$ (Kantorovitch [3], Schröder–Collatz [2]). $\leqslant$ in $\boldsymbol{R}^2$ is to be understood componentwise; superscript $^{\mathrm{T}}$ denotes transpose. We suppose that $S$ has a partial Fréchet derivative $S_y(y, w)$ with respect to $y$ for all $(y, w)$, which is equivalent to the assumption that $T$ and $f$ have partial Fréchet derivatives $T_x(x, w)$ and $f_x(x, w)$ with respect to $x$ for all $(x, w)$. Obviously,

$$(3)\qquad S_y(y, w)\Delta y = \big(T_x(x, w)\Delta x, f_x(x, w)\Delta x\big) \quad \forall \Delta y = (\Delta x, \Delta\xi) \in Y.$$

Let $y_0 = (x_0, \xi_0)$, $w_0 \in W$ satisfy the fixed point equation $y_0 = S(y_0, w_0)$. Then we call the (linearized) equation

$$(4)\qquad \Delta y = S_y(y_0, w_0)\Delta y + h$$

*variational equation corresponding to* $(y_0, w_0)$; $h$ denotes any given element $(g, \gamma) \in Y$ and $\Delta y = (\Delta x, \Delta\xi) \in Y$ a solution. The variational equation is equivalent to the following two equations

$$(5.1)\qquad \Delta x = T_x(x_0, w_0)\Delta x + g,$$

$$(5.2)\qquad \Delta\xi = f_x(x_0, w_0)\Delta x + \gamma.$$

(5.1) is called the *variational equation corresponding to* $(x_0, w_0)$. Equation (4) is uniquely solvable for each $h \in Y$ if and only if (5.1) is uniquely solvable for each $g \in X$.

In the sequel we require the existence of certain subsets $W_0 \subset W$ containing the control $w_0$ such that the derivative $S_y$ is continuous with respect to $y$ at the point $y_0$ uniformly with respect to all controls $w \in W_0$. This means that the continuity module

$$(6)\qquad \omega(r) := \omega(r; W_0, w_0, y_0) := \sup\{|||S_y(y, w) - S_y(y_0, w)||| \;\; ||y - y_0|| \leqslant r,\; w \in W_0\}$$

tends to 0 if $r$ approaches 0. Such a $W_0$ will be called a *set of varied controls for* $w_0$.

### 3. Existence of varied states, associated with varied controls

Given a pair $y_0, w_0$ satisfying $y_0 = S(y_0, w_0)$, the question arises whether it is possible to embed $y_0 = (x_0, \xi_0)$ into a family of neighbouring states $y$ corresponding to other feasible controls $w$. A partial answer is given by

THEOREM 1. *Assume that the variational equation* (4) *corresponding to* $(y_0, w_0)$ *(or equivalently,* (5.1)) *has a unique solution for each given* $h = (g, \gamma) \in Y$, *suppose that* $b$ *denotes* $||(I - T_x(x_0, w_0))^{-1}||$, $L_1$ *is an arbitrary positive number* $< 1$, $W_0$ *is a certain subset of* $W$ *such that* $w_0 \in W_0$, $\omega(\varrho) = \omega(\varrho; W_0, w_0, y_0) \to 0$ *for* $\varrho \to 0$ *and* $r$ *denotes a positive number satisfying the condition* $\omega(r) \leqslant L_1/2b$.

*Then the fixed point equation* $y = S(y, w)$ *has a solution* $y = y(w)$ *for each control* $w \in W_0$ *satisfying the restrictions*

$$(7)\qquad ||T(x_0, w) - T(x_0, w_0)|| \leqslant \frac{r(1-L_1)}{b}, \qquad ||T_x(x_0, w) - T_x(x_0, w_0)|| \leqslant \frac{L_1}{2b}.$$

*This solution is unique in* $D = \{y = (x, \xi)|\; ||x - x_0|| \leqslant r\}$ *and admits the decomposition*

$$(8)\qquad y(w) = y_0 + \Delta y + \eta,$$



*where*

$$\Delta y = [I - S_y(y_0, w_0)]^{-1}[S(y_0, w) - S(y_0, w_0)],$$

$$|\eta| = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \frac{b}{1-L_1} ||T(y_0, w) - T(y_0, w_0)||,$$

$$L_2 = \omega(r) + ||f_x(x_0, w) - f_x(x_0, w_0)|| + ||f_x(x_0, w_0)||\, L_1.$$

*Proof.* Define

$$\begin{aligned} R(y) &= R(y, w) \\ &= \big(y_0 + [I - S_y(y_0, w_0)]^{-1}[S(y, w) - S(y_0, w) - S_y(y_0, w)(y - y_0)]\big) + \\ &\quad + [I - S_y(y_0, w_0)]^{-1}[S(y_0, w) - S(y_0, w_0)] + \\ &\quad + [I - S_y(y_0, w_0)]^{-1}[S_y(y_0, w) - S_y(y_0, w_0)](y - y_0) \\ &:= R_1(y) + R_2(y) + R_3(y). \end{aligned}$$

Then $y = S(y, w)$ is equivalent to $y = R(y)$. For all $y_i = (x_i, \xi_i) \in D$ and $w \in W_0$ the first item $R_1$ of $R$ satisfies the inequality

$$\begin{aligned} &|R_1(y_1) - R_1(y_2)| \\ &= \big(||[I - T_x(x_0, w_0)]^{-1}[T(x_1, w) - T(x_2, w) - T_x(x_0, w)(x_1 - x_2)]||, \\ &\quad ||[f(x_1, w) - f(x_2, w) - f_x(x_0, w)(x_1 - x_2)] + \\ &\quad + f_x(x_0, w_0)[I - T_x(x_0, w_0)]^{-1}[T(x_1, w) - T(x_2, w) - T_x(x_0, w)(x_1 - x_2)]||^{\mathrm{T}}\big) \\ &\leqslant \big(b\omega(r), \omega(r) + ||f_x(x_0, w_0)||\, b\omega(r)\big)^{\mathrm{T}} ||x_1 - x_2||; \end{aligned}$$

the second item $R_2$ does not depend on $y$, thus $|R_2(y_1) - R_2(y_2)| = (0, 0)^{\mathrm{T}}$ and the third item $R_3$ satisfies

$$\begin{aligned} &|R_3(y_1) - R_3(y_2)| \\ &= \big(||[I - T_x(x_0, w_0)]^{-1}[T_x(x_0, w) - T_x(x_0, w_0)](x_1 - x_2)||, \\ &\quad |[f_x(x_0, w) - f_x(x_0, w_0)](x_1 - x_2) + \\ &\quad + f_x(x_0, w_0)[I - T_x(x_0, w_0)]^{-1}[T_x(x_0, w) - T_x(x_0, w_0)](x_1 - x_2)|\big)^{\mathrm{T}} \\ &\leqslant \big(b||T_x(x_0, w) - T_x(x_0, w_0)||, ||f_x(x_0, w) - f_x(x_0, w_0)|| + \\ &\quad + b||f_x(x_0, w_0)|| \cdot ||T_x(x_0, w) - T_x(x_0, w_0)||\big)^{\mathrm{T}} ||x_1 - x_2||. \end{aligned}$$

(i) If $w \in W_0$ fulfils (7), $R$ satisfies the generalized Lipschitz condition

$$|R(y_1) - R(y_2)| \leqslant L|y_1 - y_2| \qquad \forall y_1, y_2 \in D,$$

where $L$ denotes the (2, 2)-matrix $L = \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix}$.

(ii) $\sum_{k=0}^{\infty} L^k$ converges (strongly) and represents the inverse

$$(I - L)^{-1} = \frac{1}{1-L_1}\begin{bmatrix} 1 & 0 \\ L_2 & 1-L_1 \end{bmatrix}.$$

(iii) If $w \in W_0$ fulfils (7), the pseudometric ball

$$\Big\{y \in Y \mid |y-y_0| = (||x-x_0||, |\xi-\xi_0|)^{\mathrm{T}} \leqslant (I-L)^{-1}|R(y_0)-y_0| = \Big(\frac{1}{1-L_1}||[I-T_x(x_0, w_0)]^{-1}[T(x_0, w)-T(x_0, w_0)]||, \ldots\Big)^{\mathrm{T}}\Big\}$$

is contained in $D = \{y \mid ||x-x_0|| \leqslant r\}$.

Thus, all the conditions of Schröder's convergence theorem (cf. [2], § 12) for the iteration procedure $y_{k+1} = R(y_k)$ $(k = 0, 1, \ldots)$ are satisfied. The iteration procedure converges to a solution $y = y(w)$ of $y = S(y, w)$; this solution is unique in $D$, and we get the error estimation

$$|y(w)-y_1| = (I-L)^{-1} L|y_1-y_0| = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \frac{b}{1-L_1} ||T(x_0, w)-T(x_0, w_0)||.$$

Pose

$$\eta = y(w)-y_1,$$

in order to obtain the proposed decomposition $y(w) = y_0+(y_1-y_0)+\eta$. ∎

Theorem 1 shows that the solvability of the fixed point equation $y = S(y, w)$ depends on the quantities

$$||T(x_0, w)-T(x_0, w_0)||, \quad ||T_x(x_0, w)-T_x(x_0, w_0)||.$$

Beyond this, an estimation of the deviation $|y-y_0|$ involves the values

$$|f(x_0, w)-f(x_0, w_0)|, \quad ||f_x(x_0, w)-f_x(x_0, w_0)||.$$

Therefore, let $\tau(w)$ denote the quantity

$$\tau(w) = ||S(y_0, w)-S(y_0, w_0)||+||S_y(y_0, w)-S_y(y_0, w_0)||, \tag{9}$$

and let us say that *the sequence* $\{w_k\}$ *of feasible controls converges to* $w_0$ if $\tau(w_k)$ approaches 0. $\tau(w_k) \to 0$ if and only if for all four quantities

$$||T(x_0, w_k)-T(x_0, w_0)||, \quad \ldots, \quad ||f_x(x_0, w_k)-f_x(x_0, w_0)|| \to 0.$$

Theorem 1 immediately implies:

THEOREM 2. *If $W_0$ is a certain set of varied controls for $w_0$ (i.e. $w_0 \in W_0$ and $\omega(r) = \omega(r; W_0, w_0, y_0) \to 0$ for $r \to 0$) and $\{w_k\} \subset W_0$ is a sequence converging to $w_0$, then for sufficiently big $k$ there exists a solution $y_k$ of the fixed point equation $y = S(y, w_k)$, this solution $y_k$ is unique in $D_k = \{y \in Y \mid ||x-x_0|| \leqslant r_k\}$ and admits the decomposition*

$$y_k = y_0+\Delta y_k+\eta_k$$

*with*

$$\Delta y_k = [I-S_y(y_0, w_0)]^{-1}[S(y_0, w_k)-S(y_0, w_0)],$$

$$|\eta_k| \leqslant \begin{bmatrix} L_{1k} \\ L_{2k} \end{bmatrix} \frac{b}{1-L_{1k}} ||T(x_0, w_k)-T(x_0, w_0)|| \leqslant \begin{bmatrix} L_{1k} \\ L_{2k} \end{bmatrix} c||\Delta y_k||, \tag{10}$$



*where*

$$r_k = 2b||T(x_0, w_k)-T(x_0, w_0)||, \quad c = 2b||I-S_y(y_0, w_0)||,$$

$$L_{1k} = \max\{2b\omega(r_k), 2b||T_x(x_0, w_k)-T_x(x_0, w_0)||\}, \tag{11}$$

$$L_{2k} = \omega(r_k)+||f_x(x_0, w_k)-f_x(x_0, w_0)||+||f_x(x_0, w_0)||L_{1k}.$$

*Remark* 1. The phrase "for sufficiently big $k$" can be replaced by the more precise statement "for all $k$ such that $L_{1k} < \frac{1}{2}$". Obviously, $r_k$, $L_{1k}$ and $L_{2k} \to 0$ if $k \to \infty$; thus the residual part $\eta_k$ of the increment $y_k-y_0$ approaches 0 more quickly than the main part $\Delta y_k$. Note that Theorem 1 remains trivially valid, if $r = r_k$ or $L_1 = L_{1k}$ vanishes.

## 4. Directional limits

We assume $W_0$ to be a set of varied controls for $w_0$ (i.e., $\omega(r; W_0, w_0, y_0) \to 0$ for $r \to 0$) and $\{w_k\}$ to be a sequence of controls $\in W_0$ converging to $w_0$ (i.e., $\tau(w_k) \to 0$). If there is a corresponding sequence $\{\gamma_k\}$ of positive numbers approaching 0 such that the limit

$$\delta S := (\delta T, \delta f) := \lim_{k\to\infty} \frac{1}{\gamma_k}[S(y_0, w_k)-S(y_0, w_0)], \tag{12}$$

exists, we call it a *directional limit* generated by $\{w_k\}$, and its components $\delta T$, $\delta f$ *common directional limits.*

If $S$ is a directional limit generated by $\{w_k\}$, then

$$S(y_0, w_k)-S(y_0, w_0) = \gamma_k \delta S+o_1(\gamma_k), \tag{13}$$

where the correction term $o_1(\gamma_k)$ satisfies $||\gamma_k^{-1} o_1(\gamma_k)|| \to 0$ for $k \to \infty$. If $y_k$ denotes the solution of the fixed point equation $y = S(y, w_k)$ according to Theorem 2, we obtain a new (asymptotical) representation of $y_k$, namely

$$y_k = y_0 + \gamma_k[I-S_y(y_0, w_0)]^{-1} \delta S+o_2(\gamma_k), \tag{14}$$

due to (10), where again $||\gamma_k^{-1} o_2(\gamma_k)|| \to 0$ for $k \to \infty$.

*Remark* 2. In some cases, e.g. in the case of Volterra integral process equations, the directional limits $\delta T$ do not exist as elements of the original state space $X$ or as limits with respect to the norm (topology) of $X$, but they exist as elements of a bigger topological space or as limits with respect to a weaker topology (norm). In such cases we are usually concerned with the following situation:

$\gamma_k^{-1}[f(x_0, w_k)-f(x_0, w_0)]$ converges towards a $\delta f$. Though $\gamma_k^{-1}[S(y_0, w_k)-S(y_0, w_0)]$ and $\gamma_k^{-1}\Delta y_k$ (cf. (10)) does not converge in $Y = X\times\boldsymbol{R}$, it remains bounded. As a result, $\gamma_k^{-1}\eta_k$ approaches 0. $X$ is contained in a locally convex topological vector space $\tilde{X}$, whose topology induces a relative topology on $X$ weaker than the original norm topology. Thus a sequence converging in $X$ appears to be also a sequence converging with respect to the $\tilde{X}$-topology. Further, $[I-S_y(y_0, w_0)]^{-1}$ admits an extension to a linear, continuous mapping of the product space $\tilde{Y} = \tilde{X}\times\boldsymbol{R}$; this extension is denoted in the same way. An example for such an $\tilde{X}$ is the original set $X$ endowed with the weak topology. Under these hypotheses formulae (13), (14) remain valid; but the correction terms $o_i(\gamma_k)$ are now elements of $\tilde{Y}$ and $\gamma_k^{-1} o_i(\gamma_k) \to 0$ with respect to the $\tilde{Y}$-topology.

## 5. Optimality conditions

Obviously, the set $\mathfrak{S}$ of all directional limits, associated with $(y_0, w_0)$ and generated by all subsets $W_0$ and sequences $\{w_k\}$ has the property:

(i) If $\delta S \in \mathfrak{S}$, $\lambda > 0$, then $\lambda \delta S \in \mathfrak{S}$.

In all the cases we have in mind, $\mathfrak{S}$ has also the property:

(ii) If $\delta S_1$ and $\delta S_2 \in \mathfrak{S}$, then $\delta S_1 + \delta S_2 \in \mathfrak{S}$ too. As a result $\mathfrak{S}$ and also

(15) $$\mathfrak{K} = \{y \mid y = [I - S_y(y_0, w_0)]^{-1} \delta S,\ \delta S \text{ directional limit}\}$$

turn out to be convex cones.

THEOREM 3. *If the constraint set $M$ is a convex set with a nonempty interior* $\operatorname{int} M$ *and $\mathfrak{L}$ denotes*

(16) $$\mathfrak{L} = \{y \mid y = (x, \xi),\ x \in \operatorname{int} M - x_0,\ \xi < 0\},$$

*then a necessary condition for the pair $(x_0, w_0)$ to be an optimal solution is the relation $\mathfrak{K} \cap \mathfrak{L} = \emptyset$, where $y_0 = (x_0, \xi_0)$, $\xi_0 = f(x_0, w_0)$.*

*Proof.* Let $(x_0, w_0)$ be optimal. Suppose the contrary: $(m - x_0, \xi) \in \mathfrak{K} \cap \mathfrak{L}$. Then for $m \in \operatorname{int} M$, $\xi < 0$, there is a directional limit $\delta S$ such that $(m - x_0, \xi) = [I - S_y(y_0, w_0)]^{-1} \delta S$ and a convex neighbourhood $N$ of the origin such that $m + N \subset M$. Since $x_0 \in M$ and $M$ is convex, we have $(1-\gamma)x_0 + \gamma m + \gamma N \subset M$ for all $0 \leqslant \gamma \leqslant 1$. $\delta S$ is generated by a certain sequence $\{w_k\}$. To each $w_k$, $k \geqslant 1$, there corresponds a solution $y_k = S(y_k, w_k)$. Formula (14) implies

$$y_k = (x_k, \xi_k) = ((1-\gamma_k)x_0 + \gamma_k m + o_{2x}(\gamma_k),\ \xi_0 + \gamma_k \xi + o_{2\xi}(\gamma_k)),$$

where $o_{2x}$ and $o_{2\xi}$ denote the components of $o_2$. Since $\gamma_k^{-1} o_{2x}(\gamma_k) \to 0$, there is a $k_N$ such that $\gamma_k^{-1} o_{2x}(\gamma_k) \in N$ for $k \geqslant k_N$. This entails

$$x_k = T(x_k, w_k) = (1-\gamma_k)x_0 + \gamma_k m + o_{2x}(\gamma_k) \in M \qquad \forall k \geqslant k_N,$$
$$\xi_k = f(x_k, w_k) = \xi_0 + \gamma_k(\xi + \gamma_k^{-1} o_{2\xi}(\gamma_k)) < \xi_0 \qquad \forall k \geqslant 1.$$

Hence we would obtain feasible pairs $(x_k, w_k)$ with cost functional values $\xi_k < \xi_0 = f(x_0, w_0)$, which contradicts the optimality assumption concerning $(x_0, w_0)$.

*Remark* 3. In case the directional limits $\delta S$ belong only to an extended topological vector space $\tilde{Y} = \tilde{X} \times \boldsymbol{R}$ mentioned above, the convex constraint set $M$ can be considered as the intersection $\tilde{M} \cap X$ of any convex body $\tilde{M} \subset \tilde{X}$ with $X$. Define

($\tilde{16}$) $$\tilde{\mathfrak{L}} = \{y \mid y = (x, \xi),\ x \in \operatorname{int} \tilde{M} - x_0,\ \xi < 0\},$$

where the interior refers to the $\tilde{X}$-topology. Then the condition $\mathfrak{K} \cap \tilde{\mathfrak{L}} = \emptyset$ is necessary for the optimality of $(x_0, w_0)$, since $x_k \in \tilde{M} \cap X$ implies $x_k \in M$.

THEOREM 4. *If the constraint set $M$ is a convex body and the set of all directional limits $\delta S$ a convex cone, then a necessary condition for $(x_0, w_0)$ to be optimal is that there exist a nonnegative number $\varrho$ and a linear continuous functional $x^*$ over $X$ such that the variational inequality*

(17) $$(x^* + \varrho f_x(x_0, w_0))[I - T_x(x_0, w_0)]^{-1} \delta T + \varrho\, \delta f \geqslant 0 \qquad \forall\, \delta S = (\delta T, \delta f)$$

*and the maximum condition*

(18) $$x^* x_0 = \max_{x \in M} x^* x$$

*hold ($x^*$ and $\varrho$ not both zero).*



*Proof.* $\mathfrak{K}$ and $\mathfrak{L}$ (cf. (15), (16)) are convex subsets, $\mathfrak{K} \cap \mathfrak{L} = \emptyset$, $\operatorname{int} \mathfrak{L} \neq \emptyset$. Owing to a well-known separation theorem there is a number $\alpha$ and a linear continuous functional $y^*$ over $Y$, i.e., a linear, continuous functional $x^*$ over $X$ and a real number $\varrho$ such that

(i) $y^* y = x^* x + \varrho \xi \geqslant \alpha \qquad \forall y \in \mathfrak{K}$,

(ii) $y^* y \leqslant \alpha \qquad \forall y \in \mathfrak{L}$.

Because $\mathfrak{K}$ is a convex cone with vertex 0, (i) yields $0 \geqslant \alpha$. Since $x_0 \in M$ is the limit of a certain sequence $m_k \in \operatorname{int} M$, the number 0 is the limit of a sequence of negative numbers $\xi_k$ and $y_k = (m_k - x_0, \xi_k) \in \mathfrak{L}$, (ii) implies $0 \leqslant \alpha$. Therefore, $\alpha = 0$. Now on account of definition (15) of $\mathfrak{K}$ it is easy to see that (17) gives the equivalent representation of (i). Evidently, $\varrho \geqslant 0$. If $x$ denotes any element of $\operatorname{int} M$ and $\xi$ any number $< 0$, from (ii) it follows that $x^* x + \varrho \xi \leqslant x^* x_0$, and hence (18).

COROLLARY. *Define a linear continuous functional over $X$*

(19) $$\begin{aligned}\psi &= (x^* + \varrho f_x(x_0, w_0))[I - T_x(x_0, w_0)]^{-1} \\ &= ([I - T_x(x_0, w_0)]^{-1})^* (x^* + \varrho f_x(x_0, w_0)) \\ &= (I - T_x^*(x_0, w_0))^{-1}(x^* + \varrho f_x(x_0, w_0)),\end{aligned}$$

*then $\psi$ is determined by the following adjoint equation*

(20) $$\psi = T_x^*(x_0, w_0)\psi + (x^* + \varrho f_x(x_0, w_0))$$

*and the variational inequality can be written in the following way:*

(21) $$\psi\, \delta T + \varrho\, \delta f \geqslant 0 \qquad \forall (\delta T, \delta f) \in \mathfrak{S}.$$

*Remark* 4. In case the directional limits $\delta T$ exist only in an extended space $\tilde{X}$, we obtain a continuous extension of $[I - S_y(y_0, w_0)]^{-1}$ on $\tilde{Y} = \tilde{X} \times \boldsymbol{R}$ by means of a continuous extension of $[I - T_x(x_0, w_0)]^{-1}$ and of $f_x(x_0, w_0)$ onto $\tilde{X}$, which we denote by the same symbols.

We have only to pose

$$[I - S_y(y_0, w_0)]^{-1}(g, \gamma) = ([I - T_x(x_0, w_0)]^{-1} g,\ f_x(x_0, w_0)[I - T_x(x_0, w_0)]^{-1} g + \gamma)$$

for each $(g, \gamma) \in \tilde{X} \times \boldsymbol{R}$. If such continuous extensions are available and the other hypotheses of Remarks 2 and 3 are fulfilled, then propositions (17), (18) of Theorem 4 remain valid with the distinction that $x^*$ denotes a certain linear continuous functional over $\tilde{X}$ and $M$ in (18) has to be replaced by any convex body $\tilde{M} \subset \tilde{X}$ with $M = \tilde{M} \cap X$.

As to an application of a previous version of this model refer to [1].

## References


[1] L. Bittner, *On optimal control of processes governed by abstract functional equations*, Operationsforschung und Math. Statistik. 6 (1975), 1/2, pp. 107–134.

[2] L. Collatz, *Funktionalanalysis und Numerische Mathematik*, Springer-Verlag, Berlin 1964.

[3] L. V. Kantorovitch, B. Z. Vulich, A. G. Pinsker, *Functional analysis in partially ordered spaces* (in Russian), Moscow 1950.

[4] L. S. Pontriagin *et al.*, *Mathematical theory of optimal processes* (in Russian), Moscow 1961.


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# TIME OPTIMAL CONTROL PROBLEM FOR DIFFERENTIAL INCLUSIONS

V. I. BLAGODATSKIH

*Mathematical Institute of USSR Academy of Sciences, Moscow, USSR*

## 1. Introduction

Let $E^n$ be a Euclidean space of state-vectors $x = (x_1, \ldots, x_n)$ with the norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and let $\Omega(E^n)$ be the metric space of all nonempty compact subsets of $E^n$ with the Hausdorff metric

$$h(F, G) = \min\{d\colon F \subset S_d(G),\ G \subset S_d(F)\},$$

where $S_d(M)$ denotes a $d$-neighbourhood of a set $M$ in the space $E^n$.

Let us consider an object with a behaviour described by the differential inclusion

$$\dot{x} \in F(t, x), \tag{1}$$

where $F\colon E^1 \times E^n \to \Omega(E^n)$ is a given mapping. The absolutely continuous function $x(t)$ is the solution of the inclusion (1) on the interval $[t_0, t_1]$ iff the condition $\dot{x}(t) \in F\big(t, x(t)\big)$ is valid almost everywhere on this interval.

On the one hand, the differential inclusion is the extension of ordinary differential equations

$$\dot{x} = f(t, x), \tag{2}$$

when the function $f(t, x)$ is singlevalued. On the other hand, this extension is not formal: for many different problems can be transformed into differential inclusions and the development of differential inclusions permits the solution of those problems. For example, A. F. Filippov [1] investigated with the help of differential inclusions the solutions of differential equation (2) on the sets where the function $f(t, x)$ had discontinuities. N. N. Krasovski [6] used a differential inclusion for constructing a strategy in differential games. Let us consider the connection of differential inclusions with some other problems.

The optimal control problem was first considered by L. S. Pontryagin and others [8] for systems described by the equation

$$\dot{x} = f(t, x, u). \tag{3}$$