We simulated the measurements $z_i(t)$ by use of the "true" function

$$a_0(y) = .21-.28y+.7y^2.$$

The lower and upper bounds of the solution y were chosen as $\gamma_m = .3$, $\gamma_M = 2$, the interval (γ_m, γ_M) was divided into 20 intervals of length Δ, and the function $a(y)$ was represented on this interval by a continuous piecewise linear function.

To recover the function $a(y)$, we used the standard gradient method (steepest descent with projection for the case of $\mathscr{A}_{\text{ad}}$ as in (33), Franck and Wolf algorithm for the case of $\mathscr{A}_{\text{ad}}$ as in (34)).

Our numerical results are shown in figures 1 through 4.

Detailed numerical comparisons are to be found in [3].

5. Conclusion

We have given a method of computing the gradient of a functional depending on a function of the state variable and applied it to the nonlinear heat-equation.

Numerical results have been given, which show the feasibility of the method.

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A NOTE ON THE POISSON DISORDER PROBLEM

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1. Introduction

The problem can be stated roughly as follows. We observe a Poisson process N, whose rate changes from λ_0 to λ_1 (positive constants) at a certain time T. T is a random variable which is zero with probability π, or, given that $T \neq 0$, exponentially distributed with parameter λ. We want to tell when T occurred, from the observations of $\{N_t\}$. Thus the problem is to choose a stopping time τ of $\mathfrak{F}_t = \{N_s, s \leqslant t\}$ so as to minimize the expected value of some cost function depending on the difference between τ and T. Two forms of cost function are considered here; they are

$$s_\tau^1(\omega) = d(T-\tau)I_{(\tau<T)}+c(\tau-T)I_{(\tau\geqslant T)}, \tag{1.1}$$

$$s_\tau^2(\omega) = I_{(\tau<T-\varepsilon)}+c(\tau-T)I_{(\tau\geqslant T)}, \tag{1.2}$$

where ε, c, d are positive constants. It will turn out that these are special cases of a "standard problem" (see § 4). A third natural form of cost function, the "hit or miss" cost

$$s_\tau^3(\omega) = 1-I_{(T-\varepsilon\leqslant\tau\leqslant T+\varepsilon)}$$

is not standard and presents a more difficult problem.

The Wiener process version of this problem (where the observation is $N_t = \lambda(t-T)I_{(t\geqslant T)}+W_t$, $\{W_t\}$ a Wiener process) was studied by Shiryayev [5]. Shiryayev's methods were applied to the Poisson case the cost function s^2 with $\varepsilon = 0$ by Galchuk and Rozovsky [2] who with a rather complicated proof solved the problem in case $\lambda+c \geqslant \lambda_1 > \lambda_0$. Here we show that this result (Theorem 2 below) is a very simple consequence of the martingale or innovations approach to point process filtering developed in [4]. Furthermore, the solution is in fact valid for $\lambda+c \geqslant \lambda_1-\lambda_0 \geqslant 0$ and we can also obtain solutions for other cost functions such as (1.1) and (1.2) which can be rewritten in standard form.

In § 2 we state the recursive filtering result of [4], which is applied in § 3 to derive a stochastic differential equation satisfied by the process $\pi_t = P[t \geqslant T | \mathfrak{F}_t]$. In § 4 the standard problem is formulated and solved under certain conditions on the coefficients. When these conditions are not met things are more complicated and we have not been able to obtain explicit results. However, qualitatively the situation is fairly clear; some remarks on these points will be found in § 5.

2. Recursive filtering of point processes

In [4] the problem of estimating a "signal" x_t given observations of a point process $\{N_s, 0 \leqslant s \leqslant t\}$ is considered. Let $(\Omega, \mathfrak{B}, P)$ be a probability space and $\mathfrak{F}_t$ an increasing family of sub-σ-fields of $\mathfrak{B}$. All processes are assumed to be adapted to $\{\mathfrak{B}_t\}$. The signal x_t is a process of the form:

$$dx_t = f_t\,dt + dv_t, \qquad x(0) = x_0, \tag{2.1}$$

where v_t is a square-integrable martingale with respect to $\mathfrak{B}_t$ and f_t is a process satisfying

$$\mathrm{E}\int_0^t |f_s|\,ds < \infty \qquad \text{for all } t.$$

Now let λ_t be a positive, adapted process (special case: $\lambda_t = \lambda(t, x_s, s \leqslant t)$) such that

$$\mathrm{E}\int_0^t \lambda_s\,ds < \infty \qquad \text{for all } t.$$

The "observation process" N_t is a point process (piecewise constant paths, jumps of height $+1$, $N_0 = 0$) and λ_t is the "rate" of N_t, which means that $\mathrm{E}N_t < \infty$ and

$$w_t \triangleq N_t - \int_0^t \lambda_s\,ds \tag{2.2}$$

is a $\mathfrak{B}_t$-martingale. An additional assumption is that the joint quadratic variation process $\langle v, w\rangle_t$ (see [5]) is absolutely continuous with respect to Lebesgue measure, almost surely. As before, $\mathfrak{F}_t = \sigma\{N_s, \ s \leqslant t\}$.

Now let $\hat{x}_t = \mathrm{E}[x_t | \mathfrak{F}_t]$ and let $\hat{\lambda}_t$ be the predictable projection (see [1]) of λ_t on $\mathfrak{F}_t$—i.e., a predictable version of the conditional expectation $\mathrm{E}(\lambda_t | \mathfrak{F}_t)$. The following result is proved in [2]:

THEOREM 1.

(i) *The process*

$$\nu_t = N_t - \int_0^t \hat{\lambda}_s\,ds \tag{2.3}$$

is an $\mathfrak{F}_t$-martingale. This is the innovation process.

(ii) *The process $\hat{x}_t$ satisfies*

$$d\hat{x}_t = \hat{f}_t\,dt + (\hat{\lambda}_t)^{-1}\mathrm{E}\Big\{x_t(\lambda_t - \hat{\lambda}_t) + \frac{d}{dt}\langle v, w\rangle_t \,\Big|\, \mathfrak{F}_t\Big\}\,d\nu_t, \tag{2.4}$$
$$\hat{x}_0 = \mathrm{E}x_0,$$

where $\hat{f}_t = \mathrm{E}(f_t | \mathfrak{F}_t)$.

3. Formulation of the problem

We now show that the disorder problem can be put into the framework of § 2.

Let p, p^0, p^1 be independent Poisson processes with constant rates $\lambda, \lambda_0, \lambda_1$, and α a random variable independent of p, p^0, p^1 and taking values 0, 1 with probabilities π, $1-\pi$. Let $\mathfrak{B}_t = \sigma(\alpha, p_s, p_s^0, p_s^1, \ 0 \leqslant s \leqslant t)$ and T_1 be the first jump time of p. Now define

$$\begin{aligned} T &= \alpha T_1, \\ x_t &= (1-\alpha) + \alpha p_{t\wedge T}. \end{aligned} \tag{3.1}$$

Then

$$v_t = x_t - \alpha\int_0^t \lambda I_{(s<T_1)}\,ds$$

is a martingale. Since $\alpha\lambda I_{(s<T_1)} = \lambda(1-x_t)$, (3.1) can be written in the form of (2.1):

$$dx_t = \lambda(1-x_t)\,dt + dv_t, \qquad x_0 = 1-\alpha. \tag{3.2}$$

For the observations process we define

$$N_t = p^0_{t\wedge T} - (p_t^1 - p_T^1)x_t.$$

This has the properties we require and it is easily checked that $N_t - \int_0^t \lambda_s\,ds = w_t$ is a $\mathfrak{B}_t$-martingale, where

$$\lambda_t = \lambda_0(1-x_t) + \lambda_1 x_t. \tag{3.3}$$

Thus the disorder problem has the structure described in § 2. If $\pi_t = P[t \geqslant T | \mathfrak{F}_t]$, then $\pi_t = P[x_t = 1 | \mathfrak{F}_t] = \hat{x}_t$ so that the evolution of π_t is given by (2.4). We have from (3.2)

$$\hat{f}_t = \lambda(1-\pi_t).$$

The conditional distribution of x_t at time t is $x_t = 0, 1$ with probabilities $(1-\pi_t)$, π_t, so that

$$\begin{aligned} \mathrm{E}[x_t(\lambda_t - \hat{\lambda}_t) | \mathfrak{F}_t] &= (\lambda_1 - \lambda_0)\mathrm{E}[x_t(x_t - \hat{x}_t) | \mathfrak{F}_t] \\ &= (\lambda_1 - \lambda_0)\pi_t(1-\pi_t). \end{aligned}$$

Finally, $\langle v, w\rangle_t \equiv 0$ since there is zero probability that p_t and N_t jump at the same time. Thus (2.3) becomes

$$d\pi_t = \lambda(1-\pi_t)\,dt + g(\pi_{t-})\,d\nu_t, \qquad \pi_0 = \pi, \tag{3.4}$$

where

$$g(\pi_{t-}) = \frac{(\lambda_1 - \lambda_0)\pi_{t-}(1-\pi_{t-})}{\lambda_0(1-\pi_{t-}) + \lambda_1\pi_{t-}}.$$

Now

$$|g(\pi_{t-})| \leqslant \frac{|\lambda_1 - \lambda_0|}{4\min(\lambda_0, \lambda_1)},$$

so that the stochastic integral term in (3.4) is a martingale.

4. The problem

The standard problem is to find the $\mathfrak{F}_t$-stopping time τ_0 which minimizes $\mathrm{E}s_\tau^k$ where

$$s_\tau^k(\omega) = a+b\int_0^\tau (\pi_s-k)\,ds. \tag{4.1}$$

Here $a, b, k \in \mathbf{R}$, $b > 0$, $k \in [0, 1]$. Evidently, only the value of k is relevant to the minimization problem.

PROPOSITION 1.

$$\mathrm{E}s_\tau^1 = \mathrm{E}s_\tau^{k_1} \quad \textit{and} \quad \mathrm{E}s_\tau^2 = \mathrm{E}s_\tau^{k_2}$$

where

$$k_1 = d/(d+c),$$
$$k_2 = \lambda'/(\lambda'+c) \qquad (\lambda' = e^{-\varepsilon\lambda}\lambda).$$

Proof. s_τ^1 is given by (1.1). We have

$$I_{(\tau<T)}(T-\tau) = \int_\tau^\infty (1-x_s)\,ds,$$

and

$$\mathrm{E}\int_\tau^\infty (1-x_s)\,ds = \mathrm{E}\int_\tau^\infty (1-\pi_s)\,ds = \mathrm{E}\int_0^\infty (1-\pi_s)\,ds-\mathrm{E}\int_0^t (1-\pi_s)\,ds, \tag{4.2}$$

where the first expectation is finite from (3.4). Similarly,

$$\mathrm{E}I_{(\tau\geqslant T)}(\tau-T) = \mathrm{E}\int_0^\tau x_s\,ds = \mathrm{E}\int_0^\tau \pi_s\,ds; \tag{4.3}$$

combining (4.2) and (4.3) we get

$$\mathrm{E}s_\tau^1 = \mathrm{E}\int_0^\infty (1-\pi_s)\,ds+(c+d)\mathrm{E}\int_0^\tau\left(\pi_s-\frac{c}{c+d}\right)ds.$$

To calculate s^2 notice that $I_{(\tau<T-\varepsilon)} = 1-x_{\tau+\varepsilon}$ so that

$$\mathrm{E}I_{(\tau<T-\varepsilon)} = 1-\mathrm{E}(\pi_{\tau+\varepsilon}).$$

Now from (3.4), $\mathrm{E}_\pi(\pi_\varepsilon) = 1-(1-\pi)e^{-\lambda\varepsilon}$ and since π_t is a strong Markov process,

$$\mathrm{E}_{\pi_\tau}(\pi_{\tau+\varepsilon}) = (1-e^{-\lambda\varepsilon})+\pi_\tau e^{-\lambda\varepsilon}. \tag{4.4}$$

Since the last term in (3.3) is a martingale,

$$\mathrm{E}\pi_\tau = \pi+\mathrm{E}\int_0^\tau \lambda(1-\pi_s)\,ds. \tag{4.5}$$

Using (4.3)–(4.5), we finally get

$$\mathrm{E}s_\tau^2 = (1+\pi)e^{-\lambda\varepsilon}+(c+\lambda')\mathrm{E}\int_0^\tau\left(\pi_s-\frac{\lambda'}{\lambda'+c}\right)ds.$$

Thus, s^1 and s^2 reduce to the standard form, as claimed.

For the cost function s^3 we get $\mathrm{E}s_\tau^3 = 1+\mathrm{E}(x_{\tau-\varepsilon}-x_{\tau+\varepsilon})$. This cannot be reduced to the standard form because $\tau-\varepsilon$ is not a stopping time of $\mathfrak{F}_t$. Henceforth we study the standard problem and for convenience take $a = 0$, $b = 1$. From (4.1) the obvious candidate for the optimal time is

$$\tau^* = \inf\{t\colon\ \pi_t \geqslant k\}.$$

τ^* is a stopping time of $\mathfrak{F}_t$ since π_t has right-continuous paths.

PROPOSITION 2. *If τ is optimal then* $\tau \geqslant \tau^*$ a.s.

Proof. Let $A = \{\omega\colon\ \tau(\omega) < \tau^*(\omega)\}$ and suppose $PA > 0$. Then since $\pi_s < k$ for $s < \tau^*$,

$$\mathrm{E}s_{\tau\vee\tau^*} = \mathrm{E}s_\tau+\mathrm{E}I_A\int_\tau^{\tau^*} (\pi_s-k)\,ds < \mathrm{E}s_\tau.$$

Thus, $\tau\vee\tau^*$ is strictly superior to τ, so τ cannot be optimal unless $PA = 0$.

Proposition 2 can also be proved using the characteristic operator $\mathscr{A}$ of the process π_t. If 0 is the optimal stopping time for $\pi_0 = \pi$, then it is easily seen that $\mathscr{A}1(\pi) \leqslant \lambda(1-k)$; and in fact, $\mathscr{A}1(\pi) = \lambda(1-\pi)$.

The evolution of π_t (3.4) can be rewritten as

$$d\pi_t = (\lambda_1-\lambda_0)(\beta-\pi_t)(1-\pi_t)\,dt+g(\pi_{t-})\,dN_t \tag{4.6}$$

where

$$\beta = \frac{\lambda}{\lambda_1-\lambda_0}.$$

THEOREM 2. *If $\lambda_1 = \lambda_0$, or if $\lambda_1 > \lambda_0$ and $k \leqslant \beta$, then τ^* is optimal.*

Proof. Let τ be any stopping time and $B = \{w\colon\ \tau(\omega) > \tau^*(\omega)\}$. It suffices to show that $\tau\wedge\tau^*$ is superior to τ if $PB > 0$ since this combined with Proposition 2 shows that τ^* is superior to τ unless $P\{\tau = \tau^*\} = 1$.

Under the conditions stated, $\pi_t(\omega) > k$ for all $t > \tau^*(\omega)$. If $\lambda_0 = \lambda_1$ then $g \equiv 0$ and the solution to (3.3) is

$$\pi_t = 1-(1-\pi)e^{-\lambda t}$$

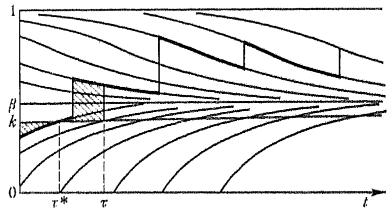
which is strictly monotonically increasing. If $\lambda_1 > \lambda_0$ then $g \geqslant 0$ so the jumps of π_t are positive. If $\beta > 1$ the sample paths of π_t are increasing. If $\beta < 1$ the solutions of (4.6) with $g \equiv 0$ are monotonic and approach β asymptotically. Hence (with $g \neq 0$) the sample path $\pi_t(\omega)$ is increasing until $t = \gamma = \inf\{s\colon \pi_s > \beta\}$ and then $\pi_t > \beta$ for all $s > \gamma$ so that in particular $\pi_t > k$ for all $t > \tau^*(\omega)$ if $k < \beta$. See Figure 1. Hence if $PB > 0$

$$\mathrm{E}s_{\tau\wedge\tau^*}^k = \mathrm{E}s_\tau^k-\mathrm{E}I_B\int_{\tau^*}^\tau (\pi_s-k)\,ds < \mathrm{E}s_\tau^k.$$

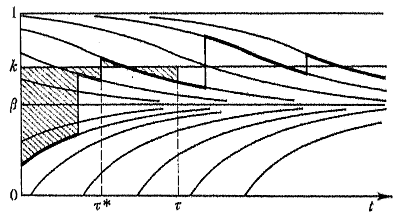
This completes the proof.

Remark. For the cost function s^2 with $\varepsilon = 0$, $k_2 < B \Leftrightarrow \lambda+c > \lambda_1-\lambda_0$. In

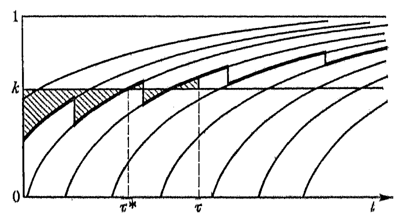
[2], Galchuk and Rozovsky obtain the result under the more restrictive conditions $\lambda+c \geqslant \lambda_1 > \lambda_0$. There is, however, an error in [2]: the expression given for the characteristic operator of the process π_t is incorrect.



Case 1: $\lambda_1 > \lambda_0$, $k \leq \beta$



Case 2: $\lambda_1 > \lambda_0$, $k > \beta$



Case 3: $\lambda_1 < \lambda_0$

Fig. 1

5. Remarks

Let us refer to the conditions of Theorem 2 as case 1; the other possibilities are $\lambda_1 > \lambda_0$ and $k > \beta$ (case 2), or $\lambda_0 > \lambda_1$ (case 3). Typical trajectories for the π_t process for the 3 cases are sketched in Figure 1. Since the quantity to be minimized is simply the expected (signed) area between the curve of π_t and level k (1), it is clear that τ^*

(1) In the figures this would be the shaded area if the process were stopped at the time τ shown.

is not in general optimal in case 2 or 3. Denote the jump times of N_t by $S_1, S_2 \ldots$; these are stopping times of $\mathfrak{F}_t$. Consider for example case 2 with $\pi = k$; then $\tau^* = 0$ but $\tau = S_1$ gives lower (not necessarily minimal) cost, and $P[S_1 > \tau^*] = 1$. It follows from Proposition 2 and results of [2] that the optimal time τ_0 is

$$\tau_0 = \inf\{t\colon \pi_t \geqslant k_0\}$$

for some $k_0 \in [k, 1]$. Since, in case 2, π_t can only enter the set $[k_0, 1]$ by jumping into it, while this never happens in case 3, we have the following:

PROPOSITION 3. *Let* $C = \bigcup_n [\tau_0 = S_n]$. *Then* $0 < PC < 1$, $PC = 1$, $PC = 0$ *in cases* 1, 2, 3, *respectively.*

However, no simple way of finding the optimal k_0 has yet been found. It involves the conditional distributions of S_1, S_2 and in conclusion we indicate how these can be derived by giving the distribution of S_1.

PROPOSITION 4. *Let $F_\pi(t)$ be the conditional distribution of S_1 given that $\pi_0 = \pi$. Then*

$$F_\pi(t) = \int_0^t \varphi(s)\,ds$$

where

$$\varphi(t) = \exp\left(\int_0^t a(s)\,ds\right) \tag{5.1}$$

and $a(s)$ is given by (5.2) *below.*

Proof. Let $T_t = E_\pi \lambda_t$. From (3.3) and (3.4), T_t is the solution of

$$\dot{T}_t = \lambda(\lambda_1 - T_t),$$
$$T_0 = (\lambda_1-\lambda_0)\pi+\lambda_0.$$

Now $N_t - \int_0^t \hat{\lambda}_s\,ds$ is an $\mathfrak{F}_t$-martingale so that

$$E_\pi(N_{t+\delta}-N_t) = E_\pi\Big[\int_t^{t+\delta} \hat{\lambda}_s\,ds\Big] = \int_t^{t+\delta} T_s\,ds.$$

Since the probability of two jumps in $[t, t+\delta]$ is $o(\delta)$, this means that

$$P_\pi[\text{jump in } [t, t+\delta]] = T_t\delta + o(\delta).$$

Also

$$P_\pi(S_1 \in [t, t+\delta]) = P_\pi(\text{no jumps in } [0, t])\cdot P_\pi(\text{jump in } [t, t+\delta]),$$

i.e.

$$F_\pi(t+\delta) - F_\pi(t) = \big(1-F_\pi(t)\big)\big(T_t\delta + o(\delta)\big).$$

Thus F_π is differentiable and

$$\varphi(t) = \Big(1-\int_0^t \varphi(s)\,ds\Big)\,T_t,$$

$$\dot{\varphi}(t) = \Big(1-\int_0^t \varphi(s)\,ds\Big)\dot{T}_t - \varphi(t)\,T_t = \varphi(t)\left(\frac{\dot{T}}{T} - T\right).$$

So $\varphi(t)$ is given by (5.1) with

$$a(t) = \frac{\dot{T}}{T} - T. \tag{5.2}$$

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OPEN-LOOP AND CLOSED-LOOP EQUILIBRIUM SOLUTIONS FOR MULTISTAGE GAMES*

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1. Introduction

In this paper we discuss a problem which arises in connection with N-player, multistage games. In particular, the so-called equilibrium solutions will be studied in detail.

Multistage games were studied earlier by several authors, e.g. Blaquière, Leitman *et al.* [1], [10]. Also Propoj in [5], [6] deals with the same type of games. But in all the works mentioned only the case of two-player, zero-sum, multistage games is considered. Very little is known about general N-player, nonzero-sum, multistage games in comparison with the existing results in the theory of differential games, e.g. see [4], [8], [9].

The following sections are partially on the author's thesis [3]. For the class of multistage games considered here we obtain necessary conditions for equilibrium solutions on the so-called *open-loop* and *closed-loop strategy classes*. Applying these conditions we derive the explicit form of the equilibrium solutions of linear multistage games with quadratic cost functionals.

2. Problem formulation and notation

In general in an N-player, nonzero-sum, multistage game we have following situation: The aim of player i, $i = 1, \ldots, N$, is to choose his control sequence (strategy) $u_0^i, u_1^i, \ldots, u_{K-1}^i$ satisfying

$$u_k^i \in U_k^i(x) = \{u^i \mid Q_k(x, u^i) = 0;\ q_k(x, u^i) \leqslant 0\}, \qquad k = 0, 1, \ldots, K-1, \tag{1}$$

to minimize his cost functional

$$J_i = g^i(x_K) + \sum_{k=0}^{K-1} h_k^i(x_k^1, \ldots, u_k^N) \tag{2}$$

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