So $\varphi(t)$ is given by (5.1) with

$$a(t) = \frac{\dot{T}}{T} - T. \tag{5.2}$$

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OPEN-LOOP AND CLOSED-LOOP EQUILIBRIUM SOLUTIONS FOR MULTISTAGE GAMES*

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1. Introduction

In this paper we discuss a problem which arises in connection with N-player, multistage games. In particular, the so-called equilibrium solutions will be studied in detail.

Multistage games were studied earlier by several authors, e.g. Blaquière, Leitman *et al.* [1], [10]. Also Propoj in [5], [6] deals with the same type of games. But in all the works mentioned only the case of two-player, zero-sum, multistage games is considered. Very little is known about general N-player, nonzero-sum, multistage games in comparison with the existing results in the theory of differential games, e.g. see [4], [8], [9].

The following sections are partially on the author's thesis [3]. For the class of multistage games considered here we obtain necessary conditions for equilibrium solutions on the so-called *open-loop* and *closed-loop strategy classes*. Applying these conditions we derive the explicit form of the equilibrium solutions of linear multistage games with quadratic cost functionals.

2. Problem formulation and notation

In general in an N-player, nonzero-sum, multistage game we have following situation: The aim of player i, $i = 1, \ldots, N$, is to choose his control sequence (strategy) $u_0^i, u_1^i, \ldots, u_{K-1}^i$ satisfying

$$u_k^i \in U_k^i(x) = \{u^i | \; Q_k(x, u^i) = 0;\; q_k(x, u^i) \leqslant 0\}, \quad k = 0, 1, \ldots, K-1, \tag{1}$$

to minimize his cost functional

$$J_i = g^i(x_K) + \sum_{k=0}^{K-1} h_k^i(x_k^1, \ldots, u_k^N) \tag{2}$$

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subject to the constraints

(3) $$x_{k+1} = f_k(x_k, u_k^1, \ldots, u_k^N), \qquad k = 0, 1, \ldots, K-1.$$

Here K is a given positive integer and $k = 0, 1, \ldots, K-1$ denotes the stage of the game. Further

$$f_k\colon E^n \times E^{m_1} \times \ldots \times E^{m_N} \to E^n, \qquad k = 0, 1, \ldots, K-1;$$

$$\left.\begin{aligned} &h_k^i\colon E^n \times E^{m_1} \times \ldots \times E^{m_N} \to E^n,\\ &Q_k^i\colon E^n \times E^{m_i} \to E^{r_k^i},\\ &q_k^i\colon E^n \times E^{m_i} \to E^{s_k^i},\end{aligned}\right\} \quad \begin{aligned} &k = 0, 1, \ldots, K-1,\\ &i = 1, \ldots, N;\end{aligned}$$

$$g^i\colon E^n \to E^1, \qquad i = 1, \ldots, N.$$

By the inequality sign for vectors in (1) we mean the following: Let $a \in E^p$; then $a \leqslant 0 \Leftrightarrow a_j \leqslant 0,\ j = 1, \ldots, p$.

We shall consider two different strategy classes defined below.

DEFINITION 1. Any sequence

$$\{u^i\} = \{u_0^i, u_1^i, \ldots, u_{K-1}^i \mid u_k^i \in U_k^i(x),\ k = 0, 1, \ldots, K-1\}$$

will denote an admissible open-loop strategy of the player i, $i = 1, \ldots, N$. An open-loop strategy N-tuple $(\{u^1\}, \ldots, \{u^N\})$ is said to be *admissible* if each $\{u^i\}$, $i = 1, \ldots, N$, is admissible.

DEFINITION 2. Any sequence

$$\{\varphi^i(x)\} = \{\varphi_0^i(x), \varphi_1^i(x), \ldots, \varphi_{K-1}^i(x) \mid \varphi_k^i(x) \in U_k^i(x),\ k = 0, 1, \ldots, K-1\},$$

where $\varphi_k^i\colon E^n \to E^{m_i}$, $k = 0, 1, \ldots, K-1$, denotes an admissible closed-loop strategy of the ith player, $i = 1, \ldots, N$. A closed-loop admissible strategy N-tuple is defined in the same way as in Definition 1.

We are interested in finding the necessary conditions for equilibrium solutions on both these strategy classes. If $(s^1, \ldots, s^N)$ denotes symbolically either an open-loop or a closed-loop strategy N-tuple, the equilibrium solution can be defined as follows.

DEFINITION 3. The admissible strategy N-tuple $(s^{*1}, \ldots, s^{*N})$ is the *equilibrium solution* (*equilibrium strategy N-tuple*) if, for $i = 1, \ldots, N$,

$$J_i^* = J_i(s^{*1}, \ldots, s^{*N}) \leqslant J_i(s^{*1}, \ldots, s^{*i-1}, s^i, s^{*i+1}, \ldots, s^{*N}),$$

where s^i is any admissible strategy of player i.

We have to make some additional assumptions about the problem just stated in order to obtain reasonable necessary conditions for both open-loop and closed-loop equilibrium solutions.

ASSUMPTION 1. *All functions appearing in* (1)–(3) *are continuously differentiable in all their arguments.*

Now write (arguments are dropped for simplicity)

$$\tilde{h}_k^i(\cdot) = h_k^i(\cdot), \qquad k = 0, 1, \ldots, K-2;\ i = 1, \ldots, N,$$
$$\tilde{h}_{K-1}^i(\cdot) = h_{K-1}^i(\cdot) + g^i(f_{K-1}(\cdot)), \qquad i = 1, \ldots, N.$$

In E^{n+1} consider the sets

$$V_k^i(x, u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^N)$$
$$= \{(a, w) \mid\ a \in E^1,\ w \in E^n;\ a = \tilde{h}_k^i(x, u^1, \ldots, u^N),\ w = f_k(x, u^1, \ldots, u^N),\ u^i \in U_k^i(x)\},$$
$$k = 0, 1, \ldots, K-1;\ i = 1, \ldots, N.$$

Further define in E^{n+1} the convex cone

$$R = \{r \mid r = (\varrho, 0, \ldots, 0),\ \varrho \leqslant 0\}.$$

We shall use also the concept of directional convexity, which we introduce for completeness.

DEFINITION 4. Denote by $R \subset E^p$ a convex cone with vertex at the origin. The set $S \subset E^p$ is said to be *R-directional convex* if, for any $x_1, x_2 \in S$ and any λ, $0 \leqslant \lambda \leqslant 1$, there exists an $x(\lambda) \in R$ such that

$$\lambda x_1 + (1-\lambda)x_2 + x(\lambda) \in S.$$

ASSUMPTION 2. *For each* $i = 1, \ldots, N$, *the sets* $V_k^i(\cdot)$, $k = 0, 1, \ldots, K-1$, *are R-directional convex for any* $x \in E^n$, $u^j \in E^{m_j}$, $j = 1, \ldots, N$, $j \neq i$.

For an admissible open-loop strategy N-tuple $(\{\hat{u}^1\}, \ldots, \{\hat{u}^N\})$ and the corresponding trajectory $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$, let us define as $I_k^i(\hat{x}_k, \hat{u}_k^i)$, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$, the set of indices belonging to the set $\{1, 2, \ldots, s_k^i\}$ and denoting those components of $q_k^i(\hat{x}_k, \hat{u}_k^i)$ for which the equality sign holds. The set $I_k^i(\hat{x}_k, \hat{u}_k^i)$ is then called the *active set* of the ith player, $i = 1, \ldots, N$, at the stage k, $k = 0, 1, \ldots, K-1$. In a similar way we also define the active set in the case of a closed-loop strategy N-tuple.

As usual, we introduce the Hamiltonian of the player i, $i = 1, \ldots, N$, at the stage k, $k = 0, 1, \ldots, K-1$, by the formula

$$H_{k+1}^i(x, u^1, \ldots, u^N) = -\tilde{h}_k^i(x, u^1, \ldots, u^N) + \lambda_{k+1}^i f_k(x, u^1, \ldots, u^N),$$

where the row-vectors $\lambda_{k+1}^i \in E^n$ will be defined later. To simplify notation in the next section, we do not explicitly write the dual variable λ_{k+1} among the arguments of H_{k+1}^i.

3. Necessary conditions for equilibrium solutions

First we shall consider equilibrium solutions on the class of open-loop strategies. From Definition 3 we can immediately conclude that, in fact, the ith player, $i = 1, \ldots, N$, has to solve only a discrete optimization problem with parameters $\{u^j\}$, $j = 1, \ldots, N$, $j \neq i$, supposing that the other players are using corresponding equilibrium strategies. So we can apply directly the discrete maximum principle which was proved in a suitable form in [3]. With some change, the formulation of the discrete maximum principle used in [2], [7] can be also applied. Thus we obtain

THEOREM 1. *Let the multistage game defined in the previous section satisfy Assumptions* 1 *and* 2. *Suppose that the admissible strategy N-tuple* $(\{u^{*1}\}, \ldots, \{u^{*N}\})$ *is the equilibrium solution of this game on the class of open-loop strategies. Further* *assume that for $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$ the vectors (indices l, m denote components of the vector constraints in* (1))

$$\frac{\partial}{\partial u_k^i} Q_{kl}^i(x_k^*, u_k^{*i}),\quad l = 1, \ldots, r_k^i;\qquad \frac{\partial}{\partial u^i} q_{km}^i(x_k^*, u_k^{*i}),\quad m \in I_k^i(x_k^*, u_k^{*i}),$$

are linearly independent. Here $x_0^*, x_1^*, \ldots, x_K^*$ denotes the equilibrium trajectory.

Then, for each $i = 1, \ldots, N$, there exist row-vectors

$$\lambda_{k+1}^i \in E^n,\quad \zeta_k^i \in E^{r_k^i},\quad \xi_k^i \in E^{s_k^i},\quad k = 0, 1, \ldots, K-1,$$

such that conditions 1)–3) *below are satisfied.*

1) $\lambda_k^i = \dfrac{\partial}{\partial x} H_{k+1}^i(x_k^*, u_k^{*1}, \ldots, u_k^{*N}) + \zeta_k^i \dfrac{\partial}{\partial x} Q_k^i(x_k^*, u_k^{*i}) + \xi_k^i \dfrac{\partial}{\partial x} q_k^i(x_k^*, u_k^{*i}),$

$k = 0, 1, \ldots, K-1$, *where we define* $\lambda_0^i = 0$ and $\lambda_K^i = -\dfrac{\partial}{\partial x} g^i(x_K^*)$.

2) $\dfrac{\partial}{\partial u_k^i} H_{k+1}^i(x, u_k^{*1}, \ldots, u_k^{*N}) + \zeta_k^i \dfrac{\partial}{\partial u_k^i} Q_k^i(x_k^*, u_k^{*i}) +$

$$+\xi_k^i \frac{\partial}{\partial u_k^i} q_k^i(x_k^*, u_k^{*i}) = 0,\qquad k = 0, 1, \ldots, K-1.$$

3) $\xi_k^i \leqslant 0$, $\xi_k^i q_k^i(x_k^*, u_k^{*i}) = 0$, $\quad k = 0, 1, \ldots, K-1$.

To formulate analogous conditions for the class of closed-loop strategies we must take into account the functional dependence given in Definition 2.

THEOREM 2. *Consider again the multistage game as in the previous theorem. Let the admissible closed-loop strategy N-tuple* $(\{\varphi^{*1}(x)\}, \ldots, \{\varphi^{*N}(x)\})$ *be the equilibrium solution of this game on the class of closed-loop strategies. Suppose that all the assumptions of Theorem* 1 *are satisfied with* $u_k^{*i} = \varphi_k^{*i}(x_k^*)$, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$, where $x_0^*, x_1^*, \ldots, x_K^*$ *denotes again the equilibrium trajectory. Moreover, let the functions* $\varphi_k^i(x)$, $k = 0, 1, \ldots, K-1$; $i = 1, \ldots, N$, *be continuously differentiable in the neighbourhood of this trajectory.*

Then, for $i = 1, \ldots, N$, there exist row-vectors

$$\lambda_{k+1}^i \in E^n,\quad \zeta_k^i \in E^{r_k^i},\quad \xi_k^i \in E^{s_k^i},\quad k = 0, 1, \ldots, K-1,$$

such that conditions 1)–3) *below are satisfied.*

1) $\lambda_k^i = \dfrac{\partial}{\partial x} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*)) +$

$$+\zeta_k^i \frac{\partial}{\partial x} Q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) + \xi_k^i \frac{\partial}{\partial x} q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) +$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{N} \left[\frac{\partial}{\partial u^j} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*))\right]\left[\frac{\partial}{\partial x} \varphi_k^{*j}(x_k^*)\right],$$

$k = 0, 1, \ldots, K-1$, *where we define* $\lambda_0^i = 0$ *and* $\lambda_K^i = -\dfrac{\partial}{\partial x} g^i(x_K^*)$.

2) $\dfrac{\partial}{\partial u^i} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*)) +$

$$+\zeta_k^i \frac{\partial}{\partial u^i} Q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) + \xi \frac{\partial}{\partial u^i} q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) = 0,$$

$$k = 0, 1, \ldots, K-1.$$

3) $\xi_k^i \leqslant 0$, $\xi_k^i q_k^i(x_k^*, \varphi_k^{*i}(x_k)) = 0$, $k = 0, 1, \ldots, K-1$.

If we compare the two theorems just stated, we see that as a rule the corresponding equilibrium costs will be different. This difference can roughly be explained in the following way. From the assumption that the players use closed-loop equilibrium strategies, i.e., they always optimize the remaining part of the process, one can conclude that certain open-loop strategy N-tuples are a priori excluded from further considerations.

On the other hand, from the definition of the open-loop equilibrium solution we see that the ith player ($i = 1, \ldots, N$) chooses his open-loop strategy while the controls of other players are assumed to be fixed. So also in this case some open-loop strategy N-tuples are not taken into account. Thus we always optimize on different sets of admissible strategies.

Moreover, from these considerations we also conclude that, in general, neither open-loop nor closed-loop equilibrium solution are necessarily noninferior (see [8], [9]), i.e., we can find in both strategy classes in question admissible strategy N-tuples which quarantee better results for all players than the equilibrium strategies.

The difficulties here considered concerning nonzero-sum multistage games are also due to the fact that meaning of optimality for general nonzero-sum games is nonunique. Thus only in the case of a two-player, zero-sum, multistage game the closed-loop equilibrium strategy of each player can be synthesised as an open-loop strategy. If $K = 1$, we have a trivial case in which both equilibria coincide because of the absence of the closed-loop effect. For a similar discussion of differential games see [8], [9].

Remark 1. It is clear that when computing the initial state x_0^* we always have N possibilities—see conditions $\lambda_0^i = 0$, $i = 1, \ldots, N$, in both theorems. These conditions must be satisfied simultaneously, because we assumed the existence of the corresponding equilibrium solution, i.e., such x_0 can be found.

Remark 2. Let the initial state x_0 be given. Both theorems remain valid with the following change: In condition 1) we always consider only $k = 1, \ldots, K$ and delete relations $\lambda_0^i = 0$, $i = 1, \ldots, N$, because they are now meaningless.

4. Linear multistage games with quadratic costs

Now we apply both theorems of the last section to a special case of multistage games, namely, to the class of linear multistage games with quadratic cost functionals. For the sake of notational simplicity we suppose that the system is autonomous, i.e., its parameters do not vary with k. The same technique is applicable to the more general cases of these games, considered in [3]. So we shall study the following multistage game (T here denotes transposition)

$$(4)\qquad x_{k+1} = Ax_k + \sum_{j=1}^{N} B_j u^j, \quad k = 0, 1, \ldots, K-1;\ K,\ x_0 \text{ given},$$

$$(5)\qquad J_i = \tfrac{1}{2}\sum_{k=0}^{K} x_k^{\mathrm{T}} Q_i X_k + \tfrac{1}{2}\sum_{k=0}^{K-1}\sum_{j=1}^{N} (u_k^j)^{\mathrm{T}} R_{ij} u_k^j, \quad i = 1, \ldots, N.$$

The dimensions of all variables are the same as in Section 2. In this way the dimensions of all matrices are also determined. Without any loss of generality we can assume that Q_i, R_{ij}, $i, j = 1, \ldots, N$, are symmetric. Moreover, let matrices Q_i, R_{ii}, $i = 1, \ldots, N$, be positive definite.

1. *Open-loop equilibrium solution.* It is evident that under the assumptions just stated the all hypotheses of Theorem 1 are satisfied. In fact, we have two equivalent ways of computing open-loop equilibrium strategies.

(A) Suppose that $(\{u^{*1}\}, \ldots, \{u^{*N}\})$ is the desired open-loop equilibrium N-tuple. As was mentioned in the last section, the ith player solves only the discrete optimization problem with parameters $\{u^{*j}\}$, $j = 1, \ldots, N$; $j \neq i$. From Theorem 1 we obtain for the ith player

$$(6)\qquad u_k^{*i} = R_{ii}^{-1} B_i^{\mathrm{T}} [P_{k+1}^i (W_k^i x_k + w_k^i) + (p_{k+1}^i)^{\mathrm{T}}], \quad k = 0, 1, \ldots, K-1,$$

where the symmetric negative definite matrices P_{k+1}^i and row-vectors p_{k+1}^i, $k = 0, 1, \ldots, K-1$, are given by the equations

$$(7)\qquad P_k^i = -Q_i + A^{\mathrm{T}}[(P_{k+1}^i)^{-1} - B_i R_{ii}^{-1} B_i^{\mathrm{T}}]^{-1} A, \quad k = 1, \ldots, K-1, \quad P_K^i = -Q_i;$$

$$(8)\qquad p_k^i = [(w_k^i)^{\mathrm{T}} P_{k+1}^i + p_{k+1}^i] A, \quad k = 1, \ldots, K-1, \quad p_K^i = 0.$$

We have used the notation ($k = 0, 1, \ldots, K-1$)

$$(9)\qquad \begin{aligned} W_k^i &= [1 - B_i R_{ii}^{-1} B_i^{\mathrm{T}} P_{k+1}^i]^{-1} A, \\ w_k^i &= [1 - B_i R_{ii}^{-1} B_i^{\mathrm{T}} P_{k+1}^i]^{-1} \Big[\sum_{\substack{j=1\\ j\neq i}}^{N} B_j u^{*j}_k + B_i R_{ii}^{-1} B_i^{\mathrm{T}} (p_{k+1}^i)^{\mathrm{T}}\Big]. \end{aligned}$$

Let us note that we have directly obtained a synthesis of the equilibrium open-loop strategies for player i. Synthesis has the same meaning here as in the theory of discrete optimal control.

Eliminating x_k from (6) by successive substitutions according to (4) with values u_k^{*i}, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$, and doing the same for each $i = 1, \ldots, N$, i.e., for each player we consider equations (6)–(9), we see that we have a system of linear algebraic equations for computing u_k^{*i}, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$.

If we consider (7) and (8) also for $k = 0$, we can write the equilibrium costs of the ith player ($i = 1, \ldots, N$) as

$$(10)\qquad J_i = -[\tfrac{1}{2} x_0^{\mathrm{T}} P_0^i x_0 + p_0^i x_0 + q_0],$$

where

$$(11)\qquad \begin{aligned} q_k^i &= q_{k+1}^i + \tfrac{1}{2}(w_k^i)^{\mathrm{T}} P_{k+1}^i w_k^i + p_{k+1}^i w_k^i - \tfrac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} (u_k^{*j})^{\mathrm{T}} R_{ij} u_k^{*j} - \\ &\quad - \tfrac{1}{2}[(w_k^i)^{\mathrm{T}} P_{k+1}^i + p_{k+1}^i] B_i R_{ii}^{-1} B_i^{\mathrm{T}} [P_{k+1}^i w_k^i + (p_{k+1}^i)^{\mathrm{T}}], \quad k = 0, 1, \ldots, K-1, \\ q_K^i &= 0. \end{aligned}$$

(B) Suppose now that each player uses synthesised open-loop strategy (6). Then we obtain

$$(12)\qquad u_k^{*i} = R_{ii}^{-1} B_i^{\mathrm{T}} \Pi_{k+1}^i \Omega_k x_k, \quad i = 1, \ldots, N;\ k = 0, 1, \ldots, K-1,$$

where, in general, non-symmetric matrices Π_{k+1}^i, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$, are obtained from the system of coupled discrete matrix Riccati equations

$$(13)\qquad \begin{aligned} \Pi_k^i &= -Q_i + \Omega_k^{\mathrm{T}} (\Pi_{k+1}^i)^{\mathrm{T}} A, \quad i = 1, \ldots, N;\ k = 1, \ldots, K-1; \\ \Pi_K^i &= -Q_i, \quad i = 1, \ldots, N, \end{aligned}$$

where

$$(14)\qquad \Omega_k = \Big[1 - \sum_{j=1}^{N} B_j R_{jj}^{-1} B_j^{\mathrm{T}} (\Pi_{k+1}^j)^{\mathrm{T}}\Big]^{-1} A, \quad k = 0, 1, \ldots, K-1.$$

In this construction we do not find an analogy of equation (10).

2. *Closed-loop equilibrium solution.* Again we suppose that this solution exists. Then we get from Theorem 2 the relations

$$(15)\qquad \varphi_k^{*i}(x) = R_{ii}^{-1} B_i^{\mathrm{T}} \tilde{P}_{k+1}^i \tilde{W}_k x, \quad i = 1, \ldots, N;\ k = 0, 1, \ldots, K-1,$$

where the symmetric matrices $\tilde{P}_{k+1}^i$, $i = 1, \ldots, N$; $k = 0, 1, \ldots, K-1$, represent again a solution of N coupled discrete matrix Riccati equations

$$(16)\qquad \begin{aligned} \tilde{P}_k^i &= -Q_i + \tilde{W}_k^{\mathrm{T}}\Big[\tilde{P}_{k+1}^i - \sum_{j=1}^{N} \tilde{P}_{k+1}^j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^{\mathrm{T}} \tilde{P}_{k+1}^j\Big] \tilde{W}_k, \\ &\qquad i = 1, \ldots, N;\ k = 1, \ldots, K-1; \\ P_K^i &= -Q_i, \quad i = 1, \ldots, N, \end{aligned}$$

where

$$(17)\qquad \tilde{W}_k = \Big[1 - \sum_{j=1}^{N} B_j R_{jj}^{-1} B_j^{\mathrm{T}} \tilde{P}_{k+1}^j\Big]^{-1} A, \quad k = 0, 1, \ldots, K-1.$$

If we formally consider equations (16) also for $k = 0$, we can compute the equilibrium costs of each player by using the formula

$$(18)\qquad \tilde{J}_i^* = -\tfrac{1}{2} x_0^{\mathrm{T}} \tilde{P}_0^i x_0, \quad i = 1, \ldots, N.$$

In general, it is not possible to find any practical conditions which would guarantee the existence of a solution of the above-mentioned system of non-homogeneous algebraic equations or the existence of all inverses in (14) and (17). So if the described constructions can be applied, we obtain in thus way the equilibrium solutions.

5. Example

We shall consider only a very simple case of a multistage game from the last section in order to be able to solve it in an explicit form. Also all variables are assumed to be scalars. Suppose that an initial state $x_0 \neq 0$ and K denoting the number of stages are given. Finally, let the three-player, multistage game be given by the equations

$$x_{k+1} = x_k + \sum_{j=1}^{3} u_k^j, \quad k = 0, 1, \ldots, K-1, \tag{19}$$

$$J_i = \tfrac{1}{2} x_K^2 + \tfrac{1}{2} \sum_{k=0}^{K-1} (u_k^i)^2, \quad i = 1, 2, 3. \tag{20}$$

Open-loop equilibrium strategies can be computed by using the construction (B) from the last section, which is more straightforward. Thus we obtain

$$\Pi_{k+1}^i = \frac{1}{3(K-k-1)+1}, \quad \Omega_k = \frac{3(K-k-1)+1}{3(K-k)+1}, \quad u_k^{*i} = -\frac{1}{3K+1}, \quad i = 1, 2, 3;\ k = 0, 1, \ldots, K-1. \tag{21}$$

Now we compute the costs of each player from (20) using equation (19)

$$J_i^* = \frac{1}{2} \cdot \frac{K+1}{(3K+1)^2} x_0^2, \quad i = 1, 2, 3. \tag{22}$$

Construction (A), i.e., solution of a system of algebraic equations, leads to the same results, as can easily be verified.

Closed-loop equilibrium strategies are obtained from relations (15)–(18). To avoid some computational difficulties we assume $K = 3$. Then we have

$$\tilde{P}_3^i = -1; \quad \tilde{P}_2^i = -\tfrac{1}{8}; \quad \tilde{P}_1^i = -\tfrac{9}{121}; \quad \tilde{P}_0^i = -\tfrac{1170}{148^2}; \quad i = 1, 2, 3,$$
$$\tilde{W}_2 = \tfrac{1}{4}; \quad \tilde{W}_1 = \tfrac{8}{11}; \quad \tilde{W}_0 = \tfrac{121}{148}.$$

From (15) we obtain

$$\begin{aligned} \varphi_0^{*i}(x) &= -\tfrac{9}{148} x = -\tfrac{9}{148} x_0, \\ \varphi_1^{*i}(x) &= -\tfrac{1}{11} x = -\tfrac{11}{148} x_0, \quad i = 1, 2, 3. \\ \varphi_2^{*i}(x) &= -\tfrac{1}{4} x = -\tfrac{22}{148} x_0, \end{aligned} \tag{23}$$

Comparing (21) and (23), we see that now the control of each player has a progressive character. From (18) we get

$$\tilde{J}_i^* = \tfrac{1}{2} \cdot \tfrac{1170}{148^2} x_0^2 = 2{,}66 \cdot 10^{-2} x_0^2, \quad i = 1, 2, 3, \tag{24}$$

while for $K = 3$ from (22) we have $J_i^* = 2 \cdot 10^{-2} x_0^2$, $i = 1, 2, 3$, which is less than in the closed-loop case.

But also in this nearly trivial multistage game we can find noninferior solutions, which are strictly superior to (22), (24), e.g. if we take

$$\hat{u}_k^i = -\frac{1}{9K+1} x_0, \quad i = 1, 2, 3; \ k = 0, 1, \ldots, K-1.$$

Then for $K = 3$ each player has the costs $1{,}78 \cdot 10^{-2} x_0^2$. Therefore, if cooperating, all players can win.

6. Conclusions

A certain class of N-player, nonzero-sum multistage games has been studied from the point of view of equilibrium solutions. We have obtained necessary conditions for the equilibrium solutions for open-loop and closed-loop strategy classes. It has been shown that these equilibria are generally different.

For a special case of linear multistage games with quadratic costs we have been able to obtain an analytic form of the equilibrium solutions, which are given by a system of discrete matrix equations of Riccati type. To illustrate the questions discussed a simple example has been computed.

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