продолжение функции $\varphi_1(x)$ *из* (14), *удовлетворяющее условиям леммы* 3, *а* $u_F(t,x)$ *удовлетворяет уравнению* (12), *однородному условию* (13) *и граничным условиям*

$$\frac{\partial^{j-1} u_F(t,0)}{\partial x^{j-1}} = f_j(t) - \frac{\partial^{j-1} u_{\varphi_0}(t,0)}{\partial x^{j-1}}, \quad j = 1, \ldots, k+1.$$

В этом случае имеет место похожий в смысле поведения по времени такой результат.

ЛЕММА 8. *Пусть функции* $\varphi_1(x)$ *и* $f_i(t)$, $i = 1, \ldots, k$, *удовлетворяют условиям леммы* 7, $h_1 > h_2 + \frac{i-1}{2k+1}$, $i = 2, \ldots, k+1$, $h_1 \geqslant h$, *и* $f_1^{(h_1)}(t) \in L_1(0,\infty)$, $f_i^{(h_2)}(t) \in L_1(0,\infty)$. *Тогда справедливо предельное равенство*

$$\lim_{t\to\infty} \bar{t}^{(h-1)} u(t,x) = \frac{f_1^{(h_1-1)}(\infty) - f_1^{(h_1-1)}(0)}{(h_1-1)!}.$$

Замечание. Построенные нами выше решения входят в класс единственности решений соответствующих задач.

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ON THE SOLUTION OF A CLASS OF EQUATIONS WITH MONOTONE OPERATORS BY ITERATION AND PROJECTION-ITERATION

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Introduction

In our lectures during the semester on "Mathematical Models and Numerical Methods" at Banach Center we were concerned with the following problems:

1. Approximative solution of equations of the type $Au = 0$, where A is a strongly monotone and Lipschitzian operator.

2. Approximative solution of problems of the type $\Lambda u + Au \ni 0$, where Λ is a (possibly multivalued) maximal monotone operator and A is again strongly monotone and Lipschitzian.

3. A posteriori error estimates for approximate solutions of equations with monotone operators.

The results we presented and further information on some special cases one can find in Gajewski–Gröger–Zacharias [11] (Kap.III.3, Kap. V), in Aubin [1] (Chap. 10) and in a series of papers of Gajewski–Gröger ([5]–[10]). Therefore, in this paper we shall not repeat the contents of our lectures. Instead of this we shall present some results, which are closely related to the subject of our lectures and which were stimulated by our stay at Banach Center. We shall consider problems of the type

$$Au + B\Lambda u \ni 0, \quad u \in D(\Lambda),$$

where Λ is maximal monotone and A, B are strongly monotone and Lipschitzian. Our results are slight generalizations of the results of Gajewski–Gröger [5] on problems of the type $\Lambda u + Au \ni 0$ mentioned above.

The paper consists of 3 sections. In Section 1 we start with a precise formulation of the problems we are interested in and we prove an existence and uniqueness result by means of the contraction principle. We show under some assumptions on A and B that it is possible to reduce the original problem to the iterative solution of a sequence of problems with linear operators instead of A and B. In Section 2 we prove the convergence of Galerkin's method and of a projection-iteration method, which combines Galerkin's method and the iteration method mentioned before. The projection-iteration method is useful especially if the maximal monotone operator Λ is linear, because in this case our problem is reduced to relatively simple linear problems. In Section 3 we consider two examples. We show that the results of Sections 1 and 2 are applicable to certain pseudo-parabolic and evolution equations. Let us remark that an existence result for initial value problems of the form

$$Au+B\frac{du}{dt} = 0, \quad u(0) = a,$$

where A and B are nonlinear operators was proved already by Barbu [2]. His assumptions on A and B are quite different from those used in this paper.

1. Iteration

Let X be a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_X$ and let X^* be the dual of X. By $\langle\cdot,\cdot\rangle$ we denote the pairing between X^* and X. The norm in the Cartesian product $X\times X$ we define by

$$\|[u,v]\|_{X\times X} = (\|u\|_X^2+\|v\|_X^2)^{1/2} \quad \forall [u,v]\in X\times X.$$

As usual (see e.g. Brezis [3]) we consider every set $\Lambda\subset X\times X$ as a multivalued mapping from X to X. We use the following notations:

$$\Lambda u = \{v|\ [u,v]\in\Lambda\} \quad \forall u\in X,$$
$$D(\Lambda) = \{u|\ u\in X, \Lambda u\neq\emptyset\}.$$

Definition 1. A set $\Lambda\subset X\times X$ is said to be *monotone* if

$$\langle u_1-u_2, v_1-v_2\rangle_X \geqslant 0 \quad \forall [u_1,v_1], \forall [u_2,v_2]\in\Lambda.$$

A monotone set $\Lambda\subset X\times X$ is said to be *maximal monotone* if it is maximal with respect to inclusion among the monotone subsets of $X\times X$.

Definition 2. If $A\in(X\to X^*)$, we define $\mathrm{Mon}(A)$ (the so-called *monotonicity constant of* A) by

$$\mathrm{Mon}(A) = \inf_{\substack{u,v\in X\\ u\neq v}} \frac{\langle Au-Av, u-v\rangle}{\|u-v\|_X^2}.$$

A is said to be (strongly) *monotone* if $\mathrm{Mon}(A)$ is (strictly) positive.

Definition 3. Let Y and Z be Banach spaces. If $A\in(Y\to Z)$, we define $\mathrm{Lip}(A)$ (the *Lipschitz constant of* A) by

$$\mathrm{Lip}(A) = \sup_{\substack{u,v\in X\\ u\neq v}} \frac{\|Au-Av\|_Z}{\|u-v\|_Y}.$$

If $\mathrm{Lip}(A)<\infty$, then A is *Lipschitzian*.

In this section we assume that we are given operators A, B and Λ such that

(1) $\quad A\in(X\to X^*), \quad m_A := \mathrm{Mon}(A)>0, \quad M_A := \mathrm{Lip}(A)<\infty,$

(2) $\quad B\in(X\to X^*), \quad m_B := \mathrm{Mon}(B)>0, \quad M_B := \mathrm{Lip}(B)<\infty,$

(3) $\quad \Lambda\subset X\times X$ is maximal monotone.

We consider the problem

$$Au+B\Lambda u\ni 0, \quad u\in D(\Lambda).$$

This problem can be written as follows:

(4) $$Au+Bv = 0, \quad [u,v]\in\Lambda.$$

We are going to formulate (4) as a fixed point problem. Let L be the (linear) duality map from X onto X^* characterized by

$$\langle Lu, v\rangle = \langle u,v\rangle_X \quad \forall u, \forall v\in X$$

and let $p>0$, $q>0$ be given real numbers. It is well known (cf. Brezis [3]) that the problem

(5) $$L(qu+pv) = q(L-pA)f+p(L-qB)g, \quad v\in\Lambda u,$$

has a unique solution $[u,v]\in X\times X$ for arbitrary $[f,g]\in X\times X$. Therefore, it makes sense to define an operator $U_{p,q}\in(X\times X\to X\times X)$ by

(6) $$U_{p,q}([qf,pg]) = [qu,pv] \Leftrightarrow [u,v] \text{ satisfies (5).}$$

Remark 1. Evidently, $[qu,pv]$ is a fixed point of $U_{p,q}$ if and only if $[u,v]$ is a solution of (4).

Lemma 1. *Let* (1)–(3) *be satisfied. If* $U_{p,q}$ *is defined by* (6) *then*

$$(\mathrm{Lip}(U_{p,q}))^2 \leqslant (\mathrm{Lip}(L-pA))^2+(\mathrm{Lip}(L-qB))^2.$$

Proof. Let $[f_i,g_i]\in X\times X$ and $[qu_i,pv_i] = U_{p,q}([qf_i,pg_i])$, $i=1,2$. Then

$$\begin{aligned}
&\|U_{p,q}([qf_1,pg_1])-U_{p,q}([qf_2,pg_2])\|_{X\times X}^2\\
&\quad= \|[q(u_1-u_2),p(v_1-v_2)]\|_{X\times X}^2 = \|q(u_1-u_2)\|_X^2+\|p(v_1-v_2)\|_X^2\\
&\quad\leqslant \|q(u_1-u_2)\|_X^2+2pq\langle u_1-u_2, v_1-v_2\rangle_X+\|p(v_1-v_2)\|_X^2\\
&\quad= \|q(u_1-u_2)+p(v_1-v_2)\|_X^2 = \|L(qu_1+pv_1)-L(qu_2+pv_2)\|_{X^*}^2\\
&\quad= \|q(L-pA)f_1+p(L-qB)g_1-q(L-pA)f_2-p(L-qB)g_2\|_{X^*}^2\\
&\quad\leqslant \big(q\,\mathrm{Lip}(L-pA)\|f_1-f_2\|_X+p\,\mathrm{Lip}(L-qB)\|g_1-g_2\|_X\big)^2\\
&\quad\leqslant \big\{(\mathrm{Lip}(L-pA))^2+(\mathrm{Lip}(L-qB))^2\big\}\big(\|q(f_1-f_2)\|_X^2+\|p(g_1-g_2)\|_X^2\big)\\
&\quad= \big\{(\mathrm{Lip}(L-pA))^2+(\mathrm{Lip}(L-qB))^2\big\}\|[qf_1,pg_1]-[qf_2,pg_2]\|_{X\times X}^2.
\end{aligned}$$

This proves the lemma.

Remark 2. Lemma 1 shows that $U_{p,q}$ is strictly contractive if

(7) $$k := (\mathrm{Lip}(L-pA))^2+(\mathrm{Lip}(L-qB))^2 < 1.$$

In view of (1) and (2) we have (see e.g. Browder–Petryshyn [4])

(8) $$(\mathrm{Lip}(L-pA))^2 \leqslant 1-2m_Ap+M_A^2p^2, \quad (\mathrm{Lip}(L-qB))^2 \leqslant 1-2m_Bq+M_B^2q^2.$$

Using (8) it is easy to see that one can satisfy (7) by a suitable choice of p and q if

$$\left(\frac{m_A}{M_A}\right)^2+\left(\frac{m_B}{M_B}\right)^2 > 1. \tag{9}$$

This relation holds e.g. if we have $A = L$ or $B = L$.

If A is a potential operator, i.e., if

$$\langle Au, v\rangle = \lim_{t\to 0}\frac{1}{t}\big(F(u+tv)-F(u)\big) \quad \forall u, \forall v \in X,$$

where $F\in(X\to \boldsymbol{R})$, then we have

$$\operatorname{Lip}(L-pA) = \max(1-m_A p, M_A p-1). \tag{10}$$

(Lemma 4.14, Kap. III, in [11] shows $\operatorname{Lip}(L-pA)\leqslant \max(1-m_A p, M_A p-1)$, and the inverse inequality can be shown easily by elementary estimations.) Correspondingly, if B is a potential operator, we have

$$\operatorname{Lip}(L-qB) = \max(1-m_B q, M_B q-1). \tag{11}$$

Therefore, in the case of potential operators A and B we can satisfy (7) by a suitable choice of p and q if

$$\frac{m_A M_A}{(m_A+M_A)^2}+\frac{m_B M_B}{(m_B+M_B)^2} > \frac{1}{4}. \tag{12}$$

THEOREM 1. *Let* (1)–(3) *be satisfied. Moreover, let A and B satisfy condition* (7) *for fixed real numbers $p > 0$ and $q > 0$. Then problem* (4) *has a unique solution* $[u, v]$. *If the sequence* $([u_i, v_i])$ *is determined by*

$$L(qu_i+pv_i) = q(L-pA)u_{i-1}+p(L-qB)v_{i-1}, \quad v_i\in \varLambda u_i,\ i = 1, 2, \ldots, \tag{13}$$
$$[u_0, v_0]\in X\times X \text{ arbitrary},$$

then $[u_i, v_i]\to[u, v]$ *in* $X\times X$; *more precisely,*

$$\|q(u_i-u)\|_X^2+\|p(v_i-v)\|_X^2 \leqslant \left(\frac{k^i}{1-k}\right)^2\big(\|q(u_1-u_0)\|_X^2+\|p(v_1-v_0)\|_X^2\big). \tag{14}$$

Proof. In view of Remark 1 the theorem is an immediate consequence of Lemma 1 and Banach's fixed point theorem.

2. Projection and projection-iteration

We assume that we are given A, B and $\varLambda$ as in Section 1. Let (X_n) be a sequence of subspaces of X such that

$$X_n\subset X_{n+1}, \quad n = 1, 2, \ldots, \quad \bigcup_n X_n \text{ is dense in } X. \tag{15}$$

We denote by X_n^* the dual space of X_n and by $\langle\cdot,\cdot\rangle_n$ the pairing between X_n^* and X_n. We define operators $A_n\in(X_n\to X_n^*)$ and $B_n\in(X_n\to X_n^*)$ by

$$\langle A_n u, v\rangle_n = \langle Au, v\rangle, \quad \langle B_n u, v\rangle_n = \langle Bu, v\rangle \quad \forall u, \forall v\in X_n. \tag{16}$$

Furthermore, we assume that $(\varLambda_n)$ is a sequence such that

$$\varLambda_n\subset X_n\times X_n \text{ is maximal monotone} \tag{17}$$

and

$$\varLambda = \{[u, v]\in X\times X \mid \lim_{n\to\infty}\inf_{[u_n, v_n]\in\varLambda_n}\|[u-u_n, v-v_n]\|_{X\times X} = 0\}. \tag{18}$$

The condition (18) means that $\varLambda$ is approximated in a certain sense by the operators $\varLambda_n$. Besides (4) we consider the corresponding "Galerkin problems"

$$A_n u_n+B_n v_n = 0, \quad [u_n, v_n]\in\varLambda_n. \tag{19}$$

THEOREM 2. *Let* (1)–(3), (7) *and* (15)–(18) *be satisfied. Then, for every n the problem* (19) *has a unique solution* $[u_n, v_n]$ *and we have*

$$[u_n, v_n]\to[u, v] \quad \text{in } X\times X, \tag{20}$$

where $[u, v]$ *denotes the solution of* (4).

Proof. Let L_n be the duality map from X_n onto X_n^* characterized by

$$\langle L_n u, v\rangle_n = \langle u, v\rangle_X \quad \forall u, \forall v\in X_n.$$

Because $\operatorname{Lip}(L_n-pA_n)\leqslant\operatorname{Lip}(L-pA)$ and $\operatorname{Lip}(L_n-qB_n)\leqslant\operatorname{Lip}(L-qB)$ the existence and uniqueness of a pair $[u_n, v_n]$ satisfying (19) follows from Theorem 1. In view of (18) there exists a sequence $([\bar u_n, \bar v_n])$ such that

$$[\bar u_n, \bar v_n]\in\varLambda_n, \quad [\bar u_n, \bar v_n]\to[u, v] \quad \text{in } X\times X. \tag{21}$$

Using (4), (19) and (21), we obtain

$$\begin{aligned}
&\|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X^2\\
&\quad= \langle q(L-pA)u_n-q(L-pA)\bar u_n+p(L-qB)v_n-p(L-qB)\bar v_n+\\
&\qquad\qquad +pq(Au-A\bar u_n+Bv-B\bar v_n),\ q(u_n-\bar u_n)+p(v_n-\bar v_n)\rangle\\
&\quad\leqslant \big\{k\big(\|q(u_n-\bar u_n)\|_X^2+\|p(v_n-\bar v_n)\|_X^2\big)^{1/2}+pq(M_A\|u-\bar u_n\|_X+M_B\|v-\bar v_n\|_X)\big\}\times\\
&\qquad\qquad\times\|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X\\
&\quad\leqslant \{k\|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X+pq(M_A\|u-\bar u_n\|_X+M_B\|v-\bar v_n\|_X)\}\times\\
&\qquad\qquad\times\|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X.
\end{aligned}$$

Hence

$$\|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X\leqslant\frac{pq}{1-k}(M_A\|u-\bar u_n\|_X+M_B\|v-\bar v_n\|_X).$$

Consequently, we have

$$\begin{aligned}
\|q(u_n-\bar u_n)\|_X^2+\|p(v_n-\bar v_n)\|_X^2 &\leqslant \|q(u_n-\bar u_n)+p(v_n-\bar v_n)\|_X^2\\
&\leqslant\left(\frac{pq}{1-k}\right)^2(M_A\|u-\bar u_n\|_X+M_B\|v-\bar v_n\|_X)^2\\
&\leqslant\frac{(pM_A)^2+(qM_B)^2}{(1-k)^2}\big(\|q(u-\bar u_n)\|_X^2+\|p(v-\bar v_n)\|_X^2\big)
\end{aligned}$$

and

(22) $$(\|q(u_n-u)\|_X^2+\|p(v_n-v)\|_X^2)^{1/2} \leqslant \left(\frac{((pM_A)^2+(qM_B)^2)}{1-k}+1\right)(\|q(u-\bar{u}_n)\|_X^2+\|p(v-\bar{v}_n)\|_X^2)^{1/2}.$$

From (21) and (22) follows assertion (20).

Remark 3. Relation (22) shows that

$$\|[u-u_n, v-v_n]\|_{X\times X} \leqslant \text{const}\cdot \inf_{[\bar{u}_n,\bar{v}_n]\in\Lambda_n} \|[u-\bar{u}_n, v-\bar{v}_n]\|_{X\times X}.$$

Therefore, the Galerkin sequence $([u_n, v_n])$ gives a "quasi-optimal" approximation of $[u, v]$ by elements of Λ_n.

The next theorem shows that under the assumptions of Theorem 2 it is possible to approximate the solution $[u, v]$ of (4) by means of a so-called projection-iteration method.

THEOREM 3. *Let* (1)–(3), (7) *and* (15)–(18) *be satisfied. If the sequence* $([\tilde{u}_n, \tilde{v}_n])$ *is determined by*

(23) $$L_n(q\tilde{u}_n+p\tilde{v}_n) = q(L_n-pA_n)\tilde{u}_{n-1}+p(L_n-qB_n)\tilde{v}_{n-1}, \quad \tilde{v}_n\in\Lambda_n\tilde{u}_n,$$
$$n = 1, 2, \ldots, \quad [\tilde{u}_0, \tilde{v}_0]\in X_1\times X_1 \textit{ arbitrary},$$

then we have

$$[\tilde{u}_n, \tilde{v}_n] \to [u, v] \quad \textit{in } X\times X,$$

where $[u, v]$ *denotes the solution of* (4).

Proof. We define $U_{p,q,n}\in(X_n\to X_n)$ by

$$U_{p,q,n}([qf, pg]) = [qu, pv] \Leftrightarrow$$
$$L_n(qu+pv) = q(L_n-pA_n)f+p(L_n-qB_n)g, \quad v\in\Lambda_n u.$$

Then $U_{p,q,n}$ is strictly contractive with the contraction constant k (cf. (7)). The fixed point of $U_{p,q,n}$ is $[qu_n, pv_n]$, where $[u_n, v_n]$ denotes the solution of (19). We can write (23) as follows:

$$[q\tilde{u}_n, p\tilde{v}_n] = U_{p,q,n}([q\tilde{u}_{n-1}, p\tilde{v}_{n-1}]), \quad n = 1, 2, \ldots, \quad [\tilde{u}_0, \tilde{v}_0]\in X_1\times X_1 \text{ arbitrary}.$$

Therefore, Theorem 3 follows immediately from Theorem 2 and Lemma 3.2, Kap. III, in [11].

3. Applications

In this section we show that it is possible to find periodic solutions of certain pseudo-parabolic equations or evolution equations using the methods of the previous sections.

Let $S = [0, T]$ be a finite interval of the real axis. If E is a Hilbert space we denote by $L^2(S; E)$ the Hilbert space of all square integrable functions defined on S with values in E (provided with the usual scalar product) and by $C(S; E)$ the space of all continuous mappings from S into E with the supremum norm.

Let V be a real Hilbert space. By V^* we denote the dual of V and by $(\cdot, \cdot)$ the pairing between V^* and V. We assume that (V_n) is a sequence of subspaces of V such that

(24) $$V_n\subset V_{n+1}, \quad n = 1, 2, \ldots, \quad \bigcup_n V_n \text{ is dense in } V.$$

By V_n^* we denote the dual of V_n and by $(\cdot,\cdot)_n$ the pairing between V_n^* and V_n. For the sake of brevity we introduce the following notations:

$$X = L^2(S; V), \quad X^* = L^2(S; V^*),$$
$$X_n = L^2(S; V_n), \quad X_n^* = L^2(S; V_n^*),$$
$$\langle f, u\rangle = \int_S (f(t), u(t))dt \quad \forall f\in X^*, \forall u\in X,$$
$$\langle f, u\rangle_n = \int_S (f(t), u(t))_n dt \quad \forall f\in X_n^*, \forall u\in X_n.$$

As in the previous section we denote by L and L_n the duality maps of X and X_n, respectively.

3.1. Pseudo-parabolic equations. If $u\in X$ we denote by u' the derivative of u in the sense of distributions on $]0, T[$ with values in V. Let

$$W = \{u|\ u\in X, u'\in X\}, \quad \|u\|_W^2 = \|u\|_X^2+\|u'\|_X^2 \quad \forall u\in W$$

and

$$W_n = \{u|\ u\in X_n, u'\in X_n\}.$$

We assume that $A\in(X\to X^*)$ and $B\in(X\to X^*)$ are operators satisfying (1) and (2). We are interested in problems of the type

(25) $$Au+Bu' = 0, \quad u\in W, \quad u(0) = u(T).$$

Such problems occur for instance in the theory of viscoelasticity, where A and B are given by elliptic differential operators.

THEOREM 4. *Let* (1), (2), (9) *and* (16) *be satisfied. Then the problem* (25) *has a unique solution* u. *If we put* $p = m_A/M_A^2$ *and* $q = m_B/M_B^2$ *and determine the sequence* (u_n) *by*

(26) $$L_n(qu_n+pu_n') = q(L_n-pA_n)u_{n-1}+p(L_n-qB_n)u_{n-1}', \quad u_n\in W_n,$$
$$u_n(0) = u_n(T), \quad n = 1, 2, \ldots, \quad u_0\in W_1 \textit{ arbitrary},$$

then we have $u_n\to u$ *in* W.

Proof. We define $\Lambda\subset X\times X$ by

$$\Lambda = \{[u, u']|\ u\in W, u(0) = u(T)\}.$$

This set Λ is maximal monotone (see e.g. [11], Lemma 1.7, Kap. VI). Evidently, problem (25) can be written as

$$Au+Bv = 0, \quad [u, v]\in\Lambda.$$

Therefore, the first part of Theorem 4 follows from Theorem 1. Let $\Lambda_n \subset X_n \times X_n$ be defined by

$$\Lambda_n = \{[u, u']|\ u \in W_n, u(0) = u(T)\}.$$

By P_n we denote the orthogonal projection from X onto X_n. If $[u, u'] \in \Lambda$ then

$$[P_n u, (P_n u)'] = [P_n u, P_n u'] \in \Lambda_n$$

and

$$[P_n u, (P_n u)'] \to [u, u'] \quad \text{in } X \times X.$$

Hence, the sequence (Λ_n) satisfies condition (18). Therefore, (26) is a formulation of method (23) in the special case considered here. Consequently, the second part of Theorem 4 follows from Theorem 3.

Remark 4. Let V_n be of finite dimension d_n and let $h_1, \ldots, h_{d_n}$ be a basis of V_n. Then we can represent u_n in the form

$$u_n = \sum_{j=1}^{d_n} c_j^n h_j, \qquad c_j^n \in L^2(S),$$

and we may regard (26) (with n fixed) as a system of linear ordinary differential equations with respect to the unknown functions c_j^n. The coefficients of this system are the elements of Gram's matrix of the basis $h_1, \ldots, h_{d_n}$, which are independent of $t \in S$.

Remark 5. In the same way as (25) we could treat initial value problems of the type

$$\text{(27)} \qquad Au + Bu' = 0, \quad u \in W, \quad u(0) = a,$$

where a is a known element of V and A, B satisfy the conditions (1), (2), (7). A somewhat different projection-iteration method for problems of the type (27) was formulated already in [11] (Kap. V). This method does not need the strong monotonicity of A and the assumption (7). On the other hand the operator that corresponds to the operator $U_{p,q}$ used here is contractive on $X = L^2(S; V)$ only, if one provides this space with the norm

$$\|u\|_{L^2,k} = \Big(\int_S \|e^{-kt}u(t)\|_V^2 dt\Big)^{1/2},$$

where k is a sufficiently great positive number. Moreover, the method requires A and B to be so-called Volterra operators.

3.2. Evolution equations. We use all notations introduced at the beginning of this section. Moreover, we assume that H and H_n, $n = 1, 2, \ldots$, are Hilbert spaces such that

$$\text{(28)} \qquad \begin{cases} V \text{ is continuously and densely imbedded into } H, \\ H_1 \subset H_2 \subset \ldots \subset H, \\ V_n \text{ is continuously and densely imbedded into } H_n. \end{cases}$$

Identifying the space H and its dual we obtain

$$V \subset H \subset V^*.$$

Similarly, we find

$$V_n \subset H_n \subset V_n^* \qquad (n = 1, 2, \ldots).$$

We denote now by u' the derivative of $u \in X$ or $u \in X_n$ in the sense of distributions on $]0, T[$ with values in V^* or V_n^*, respectively. Let

$$\tilde{W} = \{u|\ u \in X, u' \in X^*\}, \qquad \|u\|_{\tilde{W}}^2 = \|u\|_X^2 + \|u'\|_{X^*}^2 \quad \forall u \in \tilde{W}$$

and

$$\tilde{W}_n = \{u|\ u \in X_n, u' \in X_n^*\}, \qquad \|u\|_{\tilde{W}_n}^2 = \|u\|_{X_n}^2 + \|u'\|_{X_n^*}^2 \quad \forall u \in \tilde{W}_n.$$

We assume that we are given operators $A \in (X \to X^*)$ and $C \in (X^* \to X^*)$ such that A satisfies (1) and C is strongly monotone and Lipschitzian, which means that $B := CL \in (X \to X^*)$ satisfies (2). We are now interested in problems of the type

$$\text{(29)} \qquad Au + Cu' = 0, \quad u \in \tilde{W}, \quad u(0) = u(T).$$

Defining $\Lambda \subset X \times X$ by

$$\Lambda = \{[u, L^{-1}u']|\ u \in \tilde{W}, u(0) = u(T)\}$$

we can write (29) as

$$Au + Bv = 0, \quad [u, v] \in \Lambda.$$

Let

$$\Lambda_n = \{[u, L_n^{-1}u']|\ u \in W_n, u(0) = u(T)\}.$$

It is easily proved that (18) is satisfied also in this case. Therefore, it is possible to apply the results of Sections 1 and 2 to problem (29). We do not want to go into details here.

Remark 6. Gajewski and Gröger [5] have considered already two important special cases of problem (29). The first case is

$$u' + Au = 0, \quad u \in \tilde{W}, \quad u(0) = u(T),$$

and the second

$$Cu' + Lu = 0, \quad u \in \tilde{W}, \quad u(0) = u(T).$$

Gajewski and Gröger treated these cases with the aid of two different maximal monotone operators Λ whereas we use the same Λ in both cases.

Remark 7. We could apply our results also to initial value problems of the form

$$\text{(30)} \qquad Au + Cu' = 0, \quad u \in \tilde{W}, \quad u(0) = a \in H.$$

Remark 8. In the case of the problems (29) and (30) we can prove stronger results on the convergence of the methods considered in this paper, provided the operators A and C satisfy some further conditions. We shall deal with this question elsewhere (cf. Gajewski–Gröger [6]–[9], where such stronger results were proved in the two special cases mentioned in Remark 6).

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NUMERICAL METHODS FOR SOLVING VARIATIONAL INEQUALITIES

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1. Variational inequalities

Let X be a reflexive Banach space and let K be a convex, closed, non-empty subset of X. We denote by X^* the dual space of X and by $\langle\, ,\rangle$ the duality pairing between X^* and X. For a given map A which maps X into X^* we consider the following problem:

PROBLEM 1. Find $u \in K$ such that for every $v \in K$

$$\langle A(u), v-u\rangle \geqslant 0.$$

Problems of this type often arise in practice (see [1], [2]) and there is a natural need of numerical methods for solving them. Since Problem 1 is a generalization of a problem involving variational equations (for $K = X$ Problem 1 has the form of a variational equation), therefore a study of approximate methods for solving it is important for numerical methods theory.

A very detailed survey of approximate methods for solving variational inequalities is given in [1]. Here we complement the results of Mosco's paper with an estimation of the rate of convergence.

2. An approximation of a Banach space and its dual

Let Θ be a subset of the interval $(0,1]$ such that $\inf\Theta = 0$ and let n be a function mapping Θ into the set of natural numbers $\{1, 2, 3, \ldots\}$.

A family $\{X_h, p_h, r_h\}_{h\in\Theta}$ will be called an *approximation of a Banach space X* iff for every $h \in \Theta$

(1)
(i) $X_h = R^{n(h)}$, the $n(h)$-dimensional Euclidean space,
(ii) $p_h\colon X_h \to X$, p_h (prolongation) is an isomorphism from X_h onto a closed subspace P_h of X (the space of approximants),
(iii) r_h is a linear map from X into X_h which is a left inverse of p_h, i.e., for every $u_h \in X_h$ we have $r_h p_h u_h = u_h$.