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NUMERICAL METHODS FOR SOLVING VARIATIONAL INEQUALITIES

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1. Variational inequalities

Let X be a reflexive Banach space and let K be a convex, closed, non-empty subset of X. We denote by X^* the dual space of X and by $\langle\, , \rangle$ the duality pairing between X^* and X. For a given map A which maps X into X^* we consider the following problem:

PROBLEM 1. Find $u \in K$ such that for every $v \in K$

$$\langle A(u), v-u\rangle \geqslant 0.$$

Problems of this type often arise in practice (see [1], [2]) and there is a natural need of numerical methods for solving them. Since Problem 1 is a generalization of a problem involving variational equations (for $K = X$ Problem 1 has the form of a variational equation), therefore a study of approximate methods for solving it is important for numerical methods theory.

A very detailed survey of approximate methods for solving variational inequalities is given in [1]. Here we complement the results of Mosco's paper with an estimation of the rate of convergence.

2. An approximation of a Banach space and its dual

Let Θ be a subset of the interval $(0, 1]$ such that $\inf\Theta = 0$ and let n be a function mapping Θ into the set of natural numbers $\{1, 2, 3, \ldots\}$.

A family $\{X_h, p_h, r_h\}_{h\in\Theta}$ will be called an *approximation of a Banach space X* iff for every $h \in \Theta$

(1)
(i) $X_h = R^{n(h)}$, the $n(h)$-dimensional Euclidean space,
(ii) $p_h\colon X_h \to X$, p_h (prolongation) is an isomorphism from X_h onto a closed subspace P_h of X (the space of approximants),
(iii) r_h is a linear map from X into X_h which is a left inverse of p_h, i.e., for every $u_h \in X_h$ we have $r_h p_h u_h = u_h$.

Since X_h is a finite-dimensional space, then by (1ii) there exists a basis $\{\varphi_{hj}\}_{j=1}^{n(h)}$ in the space P_h such that for every $u_h = (u_h^1, \ldots, u_h^{n(h)}) \in X_h$

$$p_h u_h = \sum_{j=1}^{n(h)} u_h^j \varphi_{hj}.$$

If

$$e_{hj} = \{\delta_{kj}\}_{k=1}^{n(h)} \quad \text{for} \quad j = 1, \ldots, n(h), \qquad \delta_{kj} = \begin{cases} 1 & \text{for} \quad k = j, \\ 0 & \text{for} \quad k \neq j, \end{cases}$$

and

$$r_h = \{r_{kh}\}_{k=1}^{n(h)},$$

then by (1iii)

$$r_{kh} p_h e_{hj} = \delta_{kj} \quad \text{for} \quad k, j = 1, \ldots, n(h).$$

For every $h \in \Theta$ we define a norm in X_h putting

$$\|u_h\|_{X_h} = \|p_h u_h\|_X \quad \text{for every } u_h \in X_h.$$

Let W denote any subset of X and let

$$E_h^X(W) = \sup_{u \in W} \frac{\|p_h r_h u - u\|_X}{\|u\|_X}.$$

The approximation $\{X_h, p_h, r_h\}_{h \in \Theta}$ is convergent on the set W iff

$$\lim_{h=0} E_h^X(W) = 0.$$

We associate with an approximation $\{X_h, p_h, r_h\}_{h \in \Theta}$ of the space X a family $\{X_h^*, p_h^*, r_h^*\}_{h \in \Theta}$ defined as follows:

For every $h \in \Theta$

(i) $X_h^* = X_h = R^{n(h)}$,

(2) (ii) $p_h^* f_h = \sum_{j=1}^{n(h)} f_h^j r_{jh}$ for every $f_h = (f_h^1, \ldots, f_h^{n(h)}) \in X_h^*$,

(iii) $r_{jh}^* f = \langle f, p_h e_{hj} \rangle$ for $j = 1, \ldots, n(h)$ and for every $f \in X^*$.

LEMMA 1. *If $\{X_h, p_h, r_h\}_{h \in \Theta}$ is an approximation of a Banach space X, then the family $\{X_h^*, p_h^*, r_h^*\}_{h \in \Theta}$ is an approximation of the space X^* dual to X.*

Proof. Since $r_{1h}, \ldots, r_{n(h)h}$ are linearly independent elements of X^* ($r_{kh} p_h e_{hj} = \delta_{kj}$), therefore by (2ii) p_h^* is an isomorphism from X_h onto $P_h^* = \mathrm{Lin}(r_{1h}, \ldots, r_{n(h)h})$. Moreover, by (2iii), r_h^* is a linear map from X^* into X_h^*, and for every $f_h \in X_h^*$

$$r_h^* p_h^* f_h = \{r_{jh}^* p_h^* f_h\}_{j=1}^{n(h)} = \{\langle p_h^* f_h, p_h e_{hj} \rangle\}_{j=1}^{n(h)}$$

$$= \Big\{\sum_{k=1}^{n(h)} f_h^k \langle r_{kh}, p_h e_{hj} \rangle\Big\}_{j=1}^{n(h)} = \Big\{\sum_{k=1}^{n(h)} f_h^k \delta_{kj}\Big\}_{j=1}^{n(h)} = f_h,$$

which proves that $\{X_h^*, p_h^*, r_h^*\}_{h \in \Theta}$ is an approximation of the space X^*.

LEMMA 2. *If an approximation $\{X_h, p_h, r_h\}_{h \in \Theta}$ of a Banach space X is convergent on X, then the approximation $\{X_h^*, p_h^*, r_h^*\}_{h \in \Theta}$ of its dual X^*, defined by (2), is also convergent and*

$$E_h^{X^*}(X^*) \leqslant E_h^X(X).$$

Proof. Let us observe that for every $f \in X^*$ and $u \in X$

$$\langle p_h^* r_h^* f, u \rangle = \langle f, p_h r_h u \rangle.$$

In fact,

$$\langle p_h^* r_h^* f, u \rangle = \Big\langle \sum_{j=1}^{n(h)} (r_{jh}^* f) r_{jh}, u \Big\rangle = \sum_{j=1}^{n(h)} \langle f, p_h e_{hj} \rangle \langle r_{jh}, u \rangle$$

$$= \Big\langle f, \sum_{j=1}^{n(h)} \langle r_{jh}, u \rangle p_h e_{hj} \Big\rangle = \langle f, p_h r_h u \rangle.$$

Therefore

$$|\langle f - p_h^* r_h^* f, u \rangle| = |\langle f, u - p_h r_h u \rangle| \leqslant \|f\|_{X^*} \|u - p_h r_h u\|_X$$

and

$$E_h^{X^*}(X^*) = \sup_{f \in X^*} \frac{\|f - p_h^* r_h^* f\|_{X^*}}{\|f\|_{X^*}} = \sup_{f \in X^*} \Big\{ \frac{1}{\|f\|_{X^*}} \sup_{u \in X} \frac{|\langle f - p_h^* r_h^* f, u \rangle|}{\|u\|_X} \Big\}$$

$$\leqslant \sup_{u \in X} \frac{\|u - p_h r_h u\|_X}{\|u\|_X} = E_h^X(X),$$

which proves the assertion of Lemma 2.

3. An approximation of convex sets in a Banach space

Let X be a Banach space and let $\{X_h, p_h, r_h\}_{h \in \Theta}$ be an approximation of X. For a given non-empty, closed and convex subset K of X we define a family $\{K_h\}_{h \in \Theta}$ of sets in the following way:

(3) For every $h \in \Theta$, K_h is the closure of the set $\{u_h : u_h = r_h u, u \in K\}$ in the norm $\| \ \|_{X_h}$.

We shall give here three lemmas characterizing the family $\{K_h\}_{h \in \Theta}$.

LEMMA 3. *If K is a bounded subset of X and the family $\{K_h\}_{h \in \Theta}$ is defined by (3), then for every $h \in \Theta$, K_h is a bounded subset of X_h.*

Proof. If K is a bounded subset of X then there exists a positive constant r such that $\|v\|_X \leqslant r$ for every $v \in K$. Since for every $h \in \Theta$ and $v_h \in K_h$ there exists, by (3), a sequence $\{v^k\}_{k=1}^{\infty}$ such that for every $k = 1, \ldots, v^k \in K$ and $\lim_{k=\infty} \|v_h - r_h v^k\|_{X_h} = 0$, therefore the inequality

$$\|v_h\|_{X_h} \leqslant \|v_h - r_h v^k\|_{X_h} + \|r_h v^k\|_{X_h} \leqslant \|v_h - r_h v^k\|_{X_h} + \|p_h r_h v^k - v^k\|_X + \|v^k\|_X$$

$$\leqslant \|v_h - r_h v^k\|_{X_h} + [1 + E_h^X(K)] r,$$

which is valid for every k, $k = 1, 2, \ldots$, implies

$$\|v_h\|_{X_h} \leqslant [1+E_h^X(K)]r,$$

and this ends the proof of Lemma 3.

LEMMA 4. *If K is a closed, non-empty and convex subset of X and the family $\{K_h\}_{h\in\Theta}$ is defined by* (3), *then for every $h \in \Theta$, K_h is a closed, non-empty and convex subset of X_h.*

Proof. First we prove that for every $h \in \Theta$ K_h is a convex subset of X_h.

Let u_h, v_h be arbitrary elements from K_h. Then there exist sequences $\{u^k\}_{k=1}^{\infty}$, $\{v^k\}_{k=1}^{\infty}$ of elements which belong to K, such that

$$\lim_{k=\infty} \|u_h - r_h u^k\|_{X_h} = 0 = \lim_{k=\infty} \|v_h - r_h v^k\|_{X_h}.$$

Since K is convex, then for every $\lambda \in [0, 1]$ and $k = 1, 2, \ldots$ we have

$$w^k(\lambda) = \lambda u^k + (1-\lambda) v^k \in K$$

and

$$\lim_{k=\infty} \|[\lambda u_h + (1-\lambda)v_h] - r_h w^k(\lambda)\|_{X_h} = \lim_{k=\infty} \|\lambda(u_h - r_h u^k) + (1-\lambda)(v_h - r_h v^k)\|_{X_h} = 0,$$

which proves that $\lambda u_h + (1-\lambda) v_h \in K_h$, i.e., K_h is a convex subset of X_h.

Observing that the set K_h is non-empty by the definition of r_h (see (1)) and closed by condition (3), we end the proof of Lemma 4.

LEMMA 5. *If an approximation $\{X_h, p_h, r_h\}_{h\in\Theta}$ of a Banach space X is convergent on a subset K of X, then for every $h \in \Theta$, $h \leqslant h_1$ and for every $u_h \in K_h$*

$$\inf_{v\in K} \|v - p_h u_h\|_X \leqslant 2\|p_h u_h\|_X E_h^X(K),$$

where the family $\{K_h\}_{h\in\Theta}$ is defined by (3) *and*

$$(4)\qquad h_1 = \begin{cases} \sup\Theta, & \text{if } E_h^X(K) < \frac{1}{2} \text{ for all } h \in \Theta, \\ \inf[h|\ E_h^X(K) \geqslant \frac{1}{2}, h \in \Theta], & \text{if there exists } \tau \in \Theta \text{ such that } E_\tau^X(K) \geqslant \frac{1}{2}. \end{cases}$$

Proof. By the definition (4) of h_1 we have for every $h \in \Theta$, $h \leqslant h_1$,

$$E_h^X(K) < \tfrac{1}{2},$$

and therefore for every $v \in K$ and every $u_h \in K_h$

$$(5)\qquad \tfrac{1}{2}\|p_h u_h - v\|_X \leqslant [1 - E_h^X(K)]\|p_h u_h - v\|_X.$$

Since

$$\|p_h u_h - v\|_X \leqslant \|p_h u_h - p_h r_h v\|_X + \|p_h r_h v - v\|_X$$

and

$$\|p_h r_h v - v\|_X \leqslant E_h^X(K)\|v\|_X \leqslant E_h^X(K)[\|v - p_h u_h\|_X + \|p_h u_h\|_X],$$

then

$$(6)\qquad [1 - E_h^X(K)]\|p_h u_h - v\|_X \leqslant E_h^X(K)\|p_h u_h\|_X + \|p_h u_h - p_h r_h v\|_X.$$

Inequalities (5) and (6) imply

$$\|p_h u_h - v\|_X \leqslant 2\|p_h u_h\|_X E_h^X(K) + 2\|p_h u_h - p_h r_h v\|_X.$$

Hence we get the assertion of Lemma 5, since by (3)

$$\inf_{v\in K} \|p_h u_h - p_h r_h v\|_X = \inf_{v\in K} \|u_h - r_h v\|_{X_h} = 0.$$

4. An approximation of variational inequalities

Let $\{X_h, p_h, r_h\}_{h\in\Theta}$ be an approximation of a Banach space X and let $\{X_h^*, p_h^*, r_h^*\}_{h\in\Theta}$ be the approximation of X^*, defined by (2). For every $h \in \Theta$ we define a duality pairing between X_h^* and X_h as follows:

$$\langle f_h, u_h\rangle_h = \sum_{j=1}^{n(h)} f_h^j u_h^j$$

for every $f_h = (f_h^1, \ldots, f_h^{n(h)}) \in X_h^*$ and $u_h = (u_h^1, \ldots, u_h^{n(h)}) \in X_h$. For a given map A, which determines Problem 1 formulated in Section 1, we consider a family of operators $\{A_h\}_{h\in\Theta}$ defined by the relations:

$$(7)\qquad A_h(u_h) = r_h^* A(p_h u_h) \quad \text{for every } u_h \in X_h \text{ and } h \in \Theta.$$

We shall study the corresponding family of finite-dimensional variational inequalities:

PROBLEM 2. For every $h \in \Theta$ find $u_h \in K_h$ such that for every $v_h \in K_h$

$$\langle A_h(u_h), v_h - u_h\rangle_h \geqslant 0,$$

where the family $\{K_h\}_{h\in\Theta}$ is defined by (3) and the family $\{A_h\}_h$ is given by (7).

We shall consider properties of the family of maps $\{A_h\}_{h\in\Theta}$ which are implied by the following properties of the operation A:

(i) *strict monotonicity*: A is *strictly monotone* iff for every $u, v \in X$, $u \neq v$, we have

$$\langle A(u) - A(v), u - v\rangle > 0;$$

(ii) *strong monotonicity*: A is *strongly monotone* iff there exists a positive constant k such that for every $u, v \in X$ we have

$$k\|u - v\|_X^2 \leqslant \langle A(u) - A(v), u - v\rangle;$$

(iii) *hemicontinuity*: A is *hemicontinuous* iff for every $u, v \in X$

$$\lim_{t=+0} \langle A(u + tv), v\rangle = \langle A(u), v\rangle;$$

(iv) *coercivity*: A is *coercive on K* iff there exist an element $v_0 \in K$ and a positive constant r, such that $\|v_0\|_X < r$ and for every $v \in K$, $\|v\|_X = r$, we have

$$\langle A(v), v - v_0\rangle > 0;$$

(v) *strong coercivity*: A is *strongly coercive on* K iff there exist an element $v_0 \in K$ and positive constants r, δ, α, such that $\|v_0\|_X < r$ and for every $v \in \{x: x \in X, \operatorname{dist}(x, K) < \delta\}$ we have

$$\langle A(v), v - v_0 \rangle \geqslant \alpha.$$

The following Browder theorem justifies our interest in the properties listed above.

BROWDER'S THEOREM ([3]). *If A is a strictly monotone hemicontinuous map from a reflexive Banach space X into its dual X^*, K is a closed convex non-empty subset of X, and if either K is bounded or A is coercive on K, then there exists a unique solution of Problem* 1.

We shall prove that almost all properties from our list transfer from the operation A to the family of operations $\{A_h\}_{h\in\Theta}$.

THEOREM 1. *If A is strictly (strongly with a constant k) monotone on X, then for every $h \in \Theta$, the operation A_h defined by (7) is strictly (strongly with the constant k) monotone on X_h.*

Proof. Strict monotonicity. For every $h \in \Theta$ and every $u_h, v_h \in X_h$, $u_h \neq v_h$,

$$\langle A_h(u_h) - A_h(v_h), u_h - v_h \rangle_h = \langle A(p_h u_h) - A(p_h v_h), p_h u_h - p_h v_h \rangle > 0,$$

since $p_h u_h, p_h v_h \in X$, $p_h u_h \neq p_h v_h$ for $u_h \neq v_h$, and A is strictly monotone.

Strong monotonicity. For every $h \in \Theta$ and every $u_h, v_h \in X_h$

$$k\|u_h - v_h\|_{X_h}^2 = k\|p_h u_h - p_h v_h\|_X^2 \leqslant \langle A(p_h u_h) - A(p_h v_h), p_h u_h - p_h v_h \rangle = \langle A_h(u_h) - A_h(v_h), u_h - v_h \rangle_h,$$

since $p_h u_h, p_h v_h \in X$ and A is strongly monotone.

THEOREM 2. *If A is hemicontinuous on X, then for every $h \in \Theta$ the operation A_h defined by (7) is hemicontinuous on X_h.*

Proof. For every $h \in \Theta$ and every $u_h, v_h \in X_h$

$$\langle A_h(u_h + tv_h), v_h \rangle_h = \langle A(p_h u_h + tp_h v_h), p_h v_h \rangle.$$

Therefore, by assumed hemicontinuity of A

$$\lim_{t=+0} \langle A_h(u_h + tv_h), v_h \rangle_h = \langle A(p_h u_h), p_h v_h \rangle = \langle A_h(u_h), v_h \rangle_h.$$

THEOREM 3. *If A is bounded and strongly coercive on K and if an approximation $\{X_h, p_h, r_h\}_{h\in\Theta}$ of X is convergent on K, then there exists a positive constant h_0 such that for every $h \in \Theta$, $h < h_0$, the operation A_h defined by (7) is coercive on K_h.*

Proof. If A is strongly coercive on K, then there are given positive constants r, δ, α, and an element $v_0 \in K$ satisfying definition (v).

Let h_2 be the positive number defined by

$$h_2 = \begin{cases} \sup \Theta, & \text{if } E_h^X(K) < \delta/2r \quad \text{for all } h \in \Theta, \\ \inf [h|\ E_h^X(K) \geqslant \delta/2r, h \in \Theta], & \text{if there exists } \tau \in \Theta \text{ such that } E_\tau^X(K) \geqslant \delta/2r. \end{cases}$$

Then for every $h \in \Theta$, $h < \min[h_1, h_2]$ (h_1 defined by (4)), and for every $v_h \in K_h$, $\|v_h\|_{X_h} = r$, we have by Lemma 5

$$\operatorname{dist}(p_h v_h, K) \leqslant 2E_h^X(K)\|p_h v_h\|_X < \delta. \tag{8}$$

If h_3 is the positive number defined by

$$h_3 = \begin{cases} \sup \Theta, & \text{if } [1 + E_h^X(K)]\|v_0\|_X < r \text{ for all } h \in \Theta, \\ \inf [h|\ (1 + E_h^X(K))\|v_0\|_X \geqslant r, h \in \Theta], & \text{if there exists } \tau \in \Theta \text{ such that } [1 + E_\tau^X(K)]\|v_0\|_X \geqslant r, \end{cases}$$

then for $h < h_3$

$$\|r_h v_0\|_{X_h} = \|p_h r_h v_0\|_X \leqslant \|v_0\|_X + \|p_h r_h v_0 - v_0\|_X \leqslant [1 + E_h^X(K)]\|v_0\|_X < r. \tag{9}$$

Therefore for $h \in \Theta$, $h < \min[h_1, h_2, h_3]$, and for every $v_h \in K_h$, $\|v_h\|_{X_h} = r$, we obtain, by the assumed strong coercivity of A and by inequalities (8), (9),

$$\begin{aligned} \langle A_h(v_h), v_h - r_h v_0 \rangle_h &= \langle A(p_h v_h), p_h v_h - p_h r_h v_0 \rangle \\ &= \langle A(p_h v_h), p_h v_h - v_0 \rangle + \langle A(p_h v_h), v_0 - p_h r_h v_0 \rangle \\ &\geqslant \alpha - \|A(p_h v_h)\|_{X^*} E_h^X(K)\|v_0\|_X. \end{aligned}$$

This proves that for every $h \in \Theta$, $h < \min[h_1, h_2, h_3, h_4]$, A_h is coercive on K_h; here

$$h_4 = \begin{cases} \sup \Theta, & \text{if } \quad rE_h^X(K) \sup\limits_{\|v\|_X = r} \|A(v)\|_{X^*} < \alpha \quad \text{for all } h \in \Theta, \\ \inf [h|\ rE_h^X(K) \sup\limits_{\|v\|_X = r} \|A(v)\|_{X^*} \geqslant \alpha, h \in \Theta] & \text{otherwise.} \end{cases}$$

Theorems 1, 2, 3 enable us to formulate the following existence theorem.

THEOREM 4. *If A is a strictly monotone hemicontinuous bounded map from a reflexive Banach space X into its dual X^*, K is a closed convex non-empty subset of X and either K is bounded or A is strongly coercive on K, and if $\{X_h, p_h, r_h\}_{h\in\Theta}$ is an approximation of X convergent on K, then there exists a positive constant h_0 such that for every $h \in \Theta$, $h < h_0$, Problem 2 has a unique solution.*

Proof. The proof follows immediately from Browder's theorem, since Lemmas 3, 4 and Theorems 1, 2, 3 ensure that for $h \in \Theta$, $h < h_0$, the operation A_h and the convex subset K_h of X_h satisfy the assumptions of Browder's theorem.

5. A convergence theorem

Now we are going to prove that the family $\{u_h\}_{h\in\Theta}$ of solutions of Problem 2 is convergent to the solution of Problem 1. In the proof of this fact we shall use the following two lemmas:

LEMMA 6. *If A is a bounded strongly monotone map from a Banach space X into X^*, K is a closed convex subset of X, then*

(a) *every solution of Problem 1 is bounded,*

(b) *if* $\{X_h, p_h, r_h\}_{h\in\Theta}$ *is an approximation of* X, *convergent on* K, *solutions of Problem* 2 *are uniformly bounded.*

Proof. Since K is a closed subset of X, then there exists an element $u_0 \in K$ such that $\|u_0\|_X = \inf_{v\in K} \|v\|_X$.

For the solution u of Problem 1 we obtain from strong monotonicity of A the inequality

$$k\|u-u_0\|_X^2 \leqslant \langle A(u)-A(u_0), u-u_0\rangle \leqslant \langle A(u_0), u_0-u\rangle \leqslant \|A(u_0)\|_{X^*}\|u-u_0\|_X.$$

Consequently

$$\|u\|_X \leqslant \|u_0\|_X + \frac{1}{k}\|A(u_0)\|_X,$$

which proves assertion (a).

For the solutions u_h of Problem 2 we get by Theorem 1 an analogous inequality

$$k\|u_h-r_hu_0\|_{X_h}^2 \leqslant \|A(p_hr_hu_0)\|_{X^*}\|u_h-r_hu_0\|_{X_h};$$

therefore

$$\|u_h\|_{X_h} \leqslant \|p_hr_hu_0\|_X + \frac{1}{k}\|A(p_hr_hu_0)\|_{X^*} \quad \text{for all } h\in\Theta.$$

Since the approximation of X used here is convergent on K, then u_h are uniformly bounded.

Lemma 7. *If A is a bounded strongly monotone map from a Banach space X into X^*, K is a closed convex subset of X, $\{X_h, p_h, r_h\}_{h\in\Theta}$ is an approximation of X convergent on K, u is a solution of Problem* 1 *and u_h are solutions of Problem* 2, *then there exists a positive constant M_1 such that for every $h\in\Theta$, $h<h_1$ (h_1 defined by* (4)),

$$\langle A(u), p_hr_hu-p_hu_h\rangle \leqslant M_1E_h^X(K).$$

Proof. Since the set K is closed in X, then for every $h\in\Theta$ and every u_h which is a solution of Problem 2 there exists an element $v(h)\in K$ such that

$$\|v(h)-p_hu_h\|_X = \inf_{w\in K}\|w-p_hu_h\|_X.$$

Let u be a solution of Problem 1; then, for $h\in\Theta$, $h<h_1$,

$$\begin{aligned}\langle A(u), u-p_hu_h\rangle &= \langle A(u), u-v(h)\rangle + \langle A(u), v(h)-p_hu_h\rangle\\ &\leqslant \langle A(u), v(h)-p_hu_h\rangle \leqslant \|A(u)\|_{X^*}\|v(h)-p_hu_h\|_X\\ &\leqslant 2\|A(u)\|_{X^*}\|p_hu_h\|_X E_h^X(K) \leqslant M_2E_h^X(K),\end{aligned}$$

by Lemmas 5 and 6.

Consequently,

$$\begin{aligned}\langle A(u), p_hr_hu-p_hu_h\rangle &= \langle A(u), p_hr_hu-u\rangle + \langle A(u), u-p_hu_h\rangle\\ &\leqslant \|A(u)\|_{X^*}E_h^X(K)\|u\|_X + M_2E_h^X(K)\\ &\leqslant M_1E_h^X(K),\end{aligned}$$

which ends the proof.

Convergence theorem. *If A is a strongly monotone and λ-hölderian map from a Banach space X into X^*, K is a closed convex non-empty subset of X, $\{X_h, p_h, r_h\}_{h\in\Theta}$ is an approximation of X convergent on K, then a family of solutions of Problem* 2 *is convergent to a solution u of Problem* 1, *provided those solutions exist.*

Moreover, there exists a positive constant M such that for every $h\in\Theta$, $h<h_1$ (h_1 defined by (4)),

$$\|p_hu_h-u\|_X \leqslant M[E_h^X(K)]^\mu, \quad \text{where} \quad \mu = \min[\lambda, \tfrac{1}{2}].$$

Proof. Since A is strongly monotone, then there exists a positive constant k such that

$$\begin{aligned}k\|p_hu_h-p_hr_hu\|_X^2 &\leqslant \langle A(p_hu_h)-A(p_hr_hu), p_hu_h-p_hr_hu\rangle\\ &= \langle A_h(u_h), u_h-r_hu\rangle_h + \langle A(u)-A(p_hr_hu), p_hu_h-p_hr_hu\rangle +\\ &\quad + \langle A(u), p_hr_hu-p_hu_h\rangle.\end{aligned}$$

Taking into consideration that u_h is a solution of Problem 2 and A is hölderian, we obtain from the above inequality and from Lemma 7

$$k\|p_hu_h-p_hr_hu\|_X^2 \leqslant L\|u\|_X^\lambda[E_h^X(K)]^\lambda\|p_hu_h-p_hr_hu\|_X + M_1E_h^X(K)$$

for $h<h_1$.

Since for every y satisfying the inequality $ky^2 \leqslant ay+b$ ($k, a, b > 0$) we have $y \leqslant a/k+\sqrt{b/k}$, therefore

$$\|p_hu_h-p_hr_hu\|_X \leqslant \frac{1}{k}L\|u\|_X^\lambda[E_h^X(K)]^\lambda + \left[\frac{1}{k}M_1E_h^X(K)\right]^{1/2}$$

and by Lemma 6

$$\|u-p_hu_h\|_X \leqslant \|u-p_hr_hu\|_X + \|p_hr_hu-p_hu_h\|_X \leqslant M[E_h^X(K)]^{\min[\lambda,1/2]}.$$

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