is also the space of completely discontinuous functions that are piecewise constants, with discontinuities at the nodes $\{x_j\}_0^{N-1}$. It is not hard to show correspondingly that a natural limit when $\varepsilon = 0$ of the discrete variational formulation (5) above is a special case of the formulation of Lesaint and Raviart [2] of completely discontinuous finite element methods for ordinary differential equations.

Completely analogous results hold if we assume that $a_0 \geqslant 0$ and $a_1 < 0$. The same idea also works for variable coefficients. There is an obvious extension to problems in two dimensions if the shapes K_j are rectangles.

In a subsequent paper we establish error estimates in the maximum norm for our finite element method, which hold at each point of $\bar{\Omega}$ and which predict correctly the superconvergence results for uniform rectangular shapes of Lesaint and Raviart in the limit when $\varepsilon = 0$ for this special case.

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SOME EQUILIBRIUM AND MIXED MODELS IN THE FINITE ELEMENT METHOD

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1. Introduction

The variational formulation, used in the finite element method, is based mostly on the minimum of potential energy. As a model problem, let us consider the second order elliptic equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a_0(x)u = f, \quad x \in \Omega \subset \boldsymbol{R}^n, \tag{1.1}$$

with the following mixed boundary conditions:

$$\begin{aligned} u &= u_0 \quad \text{on } \Gamma_u, \\ a_{ij}\frac{\partial u}{\partial x_j}\nu_i &= g \quad \text{on } \Gamma_g, \\ a_{ij}\frac{\partial u}{\partial x_j}\nu_i+\alpha u &= g \quad \text{on } \Gamma_v. \end{aligned} \tag{1.1$'$}$$

Here the repeated latin index implies summation over the range 1 till n and the boundary $\partial\Omega \equiv \Gamma$ of Ω consists of four mutually disjoint parts

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \Gamma_v \cup \mathscr{R},$$

where each of Γ_u, Γ_g, Γ_v is either open in Γ or empty and the $(n-1)$-dimensional measure of $\mathscr{R}$ is zero. ν denotes the unit outward normal to Γ. Assume that the coefficients a_{ij}, a_0, α are bounded measurable functions,

$$a_0(x) \geqslant 0, \quad \alpha(x) > 0 \text{ (almost everywhere)}$$

and that a positive constant c_0 exists such that

$$a_{ij}(x)t_i t_j \geqslant c_0 t_i t_i \quad \forall t \in \boldsymbol{R}^n$$

holds almost everywhere on Ω (a.e.).

We shall consider the Sobolev spaces $W^{k,2}(\Omega)$ of functions whose derivatives up to the order k (in the sense of distributions) exist and are square-integrable in Ω. The norm in $W^{k,2}(\Omega)$ will be denoted by

$$\|u\|_k \stackrel{\mathrm{df}}{=} \Big[\sum_{|i|\leqslant k}\int_\Omega (D^i u)^2\,dx\Big]^{1/2},$$

$W^{0,2}(\Omega) = L_2(\Omega)$.

Let Ω be a bounded domain with a Lipschitz boundary. (See e.g. [1] for the definition of a L. boundary.) We introduce also the subspace of "test functions"

$$V = \{v|\ v \in W^{1,2}(\Omega),\ v = 0 \text{ on } \Gamma_u\}$$

and define the potential energy

$$\mathscr{L}(u) = \tfrac{1}{2}((u,u)) - F(u),$$

$$((u,v)) = \int_\Omega \Big(a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\Big)dx + \int_{\Gamma_v} \alpha uv\,d\Gamma,$$

$$F(u) = \int_\Omega fu\,dx + \int_{\Gamma_g\cup\Gamma_v} gu\,d\Gamma.$$

Assume that

$$f\in L_2(\Omega),\quad u_0\in W^{1,2}(\Omega),\quad g\in L_2(\Gamma_g\cup\Gamma_v).$$

According to the principle of minimum potential energy we recast the "classical" form (1.1), (1.1)′ of the problem into the following variational form:

(1.2) $\qquad \mathscr{L}(u) = \min,\quad u\in u_0+V = \{u|\ u = u_0+v,\ v\in V\}.$

(If $\Gamma_u = \emptyset$, we set $u_0 \equiv 0$.)

It is well known that if $a_0(x) \geqslant \bar{a}_0 > 0$ a.e. or if $\Gamma_u \cup \Gamma_v = \emptyset$ with $\alpha(x) \geqslant \bar{\alpha} > 0$, there exists precisely one minimizing element u. Moreover, it holds

$$((u,v)) = F(v) \qquad \forall v\in V,$$

and u is reffered to as a *weak solution* of (1.1), (1.1)′.

Sometimes we are more interested in the *cogradient-vector*

$$\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),\qquad i = 1,\ldots,n,$$

than in the solution u itself (stress components in the torsion problem, heat flow, velocity in the seepage a.s.o.). If this is the case, we may proceed in three ways:

(i) to find an approximate solution of the problem (1.2) and to evaluate the derivatives (Compatible models of finite elements);

(ii) to apply a dual variational formulation, called the principle of minimum complementary energy, which yields an approximate cogradient directly (Equilibrium models);

(iii) to apply a mixed variational formulation (based on a principle of Hellinger–Reissner type), enabling us to obtain approximations to both the solution and its cogradient simultaneously (Hybrid or Mixed models).

In the sequel we discuss the equilibrium models and a particular mixed model, deriving some error estimates.

In Section 2 the case of a strictly positive coefficient a_0 is considered, when the subspaces of "equilibriated" finite elements are easy to construct and the convergence theorems are based on some results of J.-P. Aubin and H. Burchard [2].

Section 3 is devoted to the case $a_0 \equiv 0$. Using piecewise linear polynomials on a triangulation, the convergence of order $O(h^2)$ is proved by means of a suitable projection operator.

In Section 4 we establish a kind of a mixed model, which is based on a new variational formulation of the Dirichlet boundary value problem. An error estimate is also derived.

2. Equation with an absolute term

In this section we assume that an "absolute term" $a_0 u$ occurs in equation (1.1), the coefficient a_0 being strictly positive, i.e.,

$$a_0(x) \geqslant \bar{a}_0 > 0 \quad \text{a.e. in } \Omega.$$

First we derive the functional of complementary energy by means of the Friedrichs transform (see [3]).

Define

(2.1) $$\eta_i = \frac{\partial u}{\partial x_i},\quad i = 1,\ldots,n,\quad \eta_{n+1} = u$$

and rewrite the problem (1.2) in the form

$$\mathscr{L}(u) = \mathscr{L}_1(u;\eta_1,\ldots,\eta_{n+1})$$
$$= \tfrac{1}{2}\int_\Omega (a_{ij}\eta_i\eta_j + a_0\eta_{n+1}^2)dx + \tfrac{1}{2}\int_{\Gamma_v}\alpha u^2 d\Gamma - \int_\Omega fu\,dx - \int_{\Gamma_g\cup\Gamma_v} gu\,d\Gamma.$$

Let us regard conditions (2.1) and the boundary condition on Γ_u to $\mathscr{L}_1$ jointly as side-conditions (constraints) by means of Lagrange multipliers $\lambda_i(x)$, $\mu(x)$. Thus we obtain the functional

$$\mathscr{H}(u,\eta_j,\lambda_j,\mu_j)$$
$$= \mathscr{L}_1(u,\eta_j) + \int_\Omega \Big[\lambda_i\Big(\frac{\partial u}{\partial x_i} - \eta_i\Big) + \lambda_{n+1}(u - \eta_{n+1})\Big]dx + \int_{\Gamma_u}\mu(u-u_0)\,d\Gamma.$$

Integration by parts yields

$$\int_\Omega \lambda_i\frac{\partial u}{\partial x_i}dx = -\int_\Omega u\,\mathrm{div}\,\lambda\,dx + \int_\Gamma \lambda_i\nu_i u\,d\Gamma,$$

where

$$\mathrm{div}\,\boldsymbol{\lambda} = \frac{\partial\lambda_i}{\partial x_i}.$$

Taking the variations with respect to η_i, u only, we obtain

$$\delta_{\eta_j, u}\mathscr{H} = \int_\Omega (a_{ij}\eta_i\,\delta\eta_j + a_0\eta_{n+1}\,\delta\eta_{n+1})\,dx +$$

$$+\int_{\Gamma_v} \alpha u\,\delta u\,d\Gamma - \int_\Omega f\,\delta u\,dx - \int_{\Gamma_g\cup\Gamma_v} g\,\delta u\,d\Gamma +$$

$$+\int_\Omega [-\lambda_i\,\delta\eta_i - \lambda_{n+1}\,\delta\eta_{n+1} - \operatorname{div}\lambda\cdot\delta u + \lambda_{n+1}\,\delta u]\,dx +$$

$$+\int_{\Gamma_u} \mu\,\delta u\,d\Gamma + \int_\Gamma \lambda_i\nu_i\,\delta u\,d\Gamma.$$

Setting

$$\delta_{\eta_j, u}\mathscr{H} = 0,$$

we deduce the following conditions:

$$a_{ij}\eta_i = \lambda_j, \quad j = 1, \ldots, n, \tag{2.2}$$

$$a_0\eta_{n+1} = \lambda_{n+1} \quad \text{in } \Omega,$$

$$\lambda_{n+1} = \operatorname{div}\lambda + f \quad \text{in } \Omega, \tag{2.3}$$

$$\lambda_i\nu_i = g \quad \text{on } \Gamma_g, \tag{2.4}$$

$$\lambda_i\nu_i + \alpha u = g \quad \text{on } \Gamma_v, \tag{2.5}$$

$$\lambda_i\nu_i + \mu = 0 \quad \text{on } \Gamma_u. \tag{2.6}$$

Eliminating η_j, u, μ from the functional $\mathscr{H}$ by means of (2.2) till (2.6), we are led to the functional

$$\mathscr{S}_1(\lambda_1, \ldots, \lambda_{n+1}) = -\tfrac{1}{2}\int_\Omega (b_{ij}\lambda_i\lambda_j + b_0\lambda_{n+1}^2)\,dx - \tag{2.7}$$

$$-\tfrac{1}{2}\int_{\Gamma_v} \alpha^{-1}(\lambda_i\nu_i - g)^2\,d\Gamma + \int_{\Gamma_u} \lambda_i\nu_i u_0\,d\Gamma,$$

where b_{ij} are the entries of the matrix $[a]^{-1}$, $b_0 = a_0^{-1}$.

The functional $\mathscr{S}_1$ has to be considered with the side-conditions (constraints) (2.3), (2.4). Condition (2.3), however, suggests to eliminate also λ_{n+1}, which yields a new functional

$$\mathscr{S}_2(\lambda_1, \ldots, \lambda_n) = -\tfrac{1}{2}\int_\Omega [b_{ij}\lambda_i\lambda_j + b_0(\operatorname{div}\lambda + f)^2]\,dx - \tag{2.8}$$

$$-\tfrac{1}{2}\int_{\Gamma_v} \alpha^{-1}(\lambda_i\nu_i - g)^2\,d\Gamma + \int_{\Gamma_u} \lambda_i\nu_i u_0\,d\Gamma.$$

According to the general concept of the Friedrichs transform, we may expect that

$$\mathscr{S}_2 = \max \text{ over } \Lambda \Leftrightarrow \lambda_i = a_{ij}\frac{\partial u}{\partial x_j} \quad (i = 1, \ldots, n),$$

where Λ is an appropriate set of admissible vector-functions λ, satisfying (2.4) and u is the solution of the problem (1.2).

Let us verify the latter conjecture. To this end, we shall distinguish two cases: (i) $\Gamma_v = \emptyset$, (ii) $\Gamma_v \neq \emptyset$. Here we study only the easier case (i); the case (ii) requires some more mathematics, which can be found in the paper [4].

Henceforth we assume that $\Gamma_v = \emptyset$. Define

$$H = [L_2(\Omega)]^{n+1},$$

and the bilinear form

$$(\lambda', \lambda'')_H = \int_\Omega (b_{ij}\lambda_i'\lambda_j'' + b_0\lambda_{n+1}'\lambda_{n+1}'')\,dx \tag{2.9}$$

on $H\times H$. The coefficients b_{ij}, b_0 are bounded and measurable. Moreover, $b_{ij} = b_{ji}$ and there exists a positive constant c_1 such that

$$\|\lambda\|_H^2 = (\lambda, \lambda)_H \geqslant c_1 \sum_{i=1}^{n+1} \|\lambda_i\|_0^2.$$

Consequently, (2.9) is a scalar product and H with (2.9) is a Hilbert space.

Definition 2.1. Define

$$B(\lambda, v) = \int_\Omega \left(\lambda_i\frac{\partial v}{\partial x_i} + \lambda_{n+1}v\right)dx \quad \forall\lambda \in H,\ v \in W^{1,2}(\Omega),$$

$$H_1 = \left\{\begin{array}{l}\lambda \in H \mid \exists v \in V,\ \lambda_i = a_{ij}\dfrac{\partial v}{\partial x_j} \quad (i = 1, \ldots, n) \\ \qquad\qquad \lambda_{n+1} = a_0 v\end{array}\right\} \equiv \lambda = \lambda(v),$$

$$H_2 = \{\lambda \in H \mid B(\lambda, v) = 0 \ \forall v \in V\},$$

$$\Lambda_{f,g} = \{\lambda \in H \mid B(\lambda, v) = F(v) \ \forall v \in V\}.$$

Remark 2.1. We can give a mechanical interpretation of the sets introduced above:

H_1—stresses compatible with virtual displacements,

H_2—virtual stresses,

$\Lambda_{f,g}$—statically admissible stresses.

Any vector $\lambda \in \Lambda_{f,g}$ satisfies (2.3), (2.4) in the weak sense (in the sense of the principle of virtual displacements).

Theorem 2.1 (Principle of minimum complementary energy). *Define*

$$\mathscr{S}(\lambda) = \tfrac{1}{2}\|\lambda - \lambda(u_0)\|_H^2.$$

Then

$$\mathscr{S}(\lambda) = \min, \quad \lambda \in \Lambda_{f,g}$$

if and only if

$$\lambda = \lambda(u),$$

where u is the solution of the problem (1.2).

It holds

$$-\mathscr{S}(\lambda(u)) = \mathscr{L}(u)+F(u_0)-\tfrac{1}{2}\|\lambda(u_0)\|_H^2 .$$

Proof. The proof is based on the fact that H_1 is orthogonal to H_2. In fact, let $\lambda' \in H_1$, $\lambda'' \in H_2$. Then we have

$$(\lambda, \lambda'')_H = \int_\Omega \left[b_{ij}\left(a_{ik}\frac{\partial v}{\partial x_k}\right)\lambda''_j + b_0 a_0 v \lambda''_{n+1}\right] dx$$

$$= \int_\Omega \left(\frac{\partial v}{\partial x_j}\lambda''_j + v\lambda''_{n+1}\right) dx = B(\lambda'', v) = 0.$$

Next let us set $u = u_0+w$, $w \in V$. Then $\lambda(w) \in H_1$. Secondly, $\lambda-\lambda(u) \in H_2$ for $\lambda \in \Lambda_{f,g}$. In fact, note that

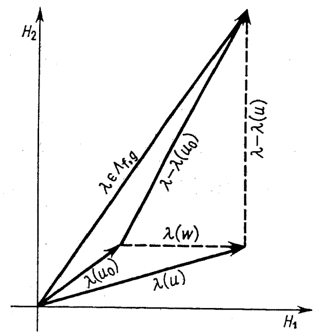
$$B(\lambda(u), v) = ((u, v)) = F(v) \quad \forall v \in V.$$

Consequently, for $\lambda \in \Lambda_{f,g}$ we may write

$$\|\lambda-\lambda(u_0)\|_H^2 = \|\lambda-\lambda(u)+\lambda(u)-\lambda(u_0)\|_H^2 = \|\lambda-\lambda(u)\|_H^2+\|\lambda(w)\|_H^2$$

and the assertion of the theorem follows.

Remark 2.2. We can prove that H_1 and H_2 are closed subspaces of H and $H = H_1 \oplus H_2$. Thus the principle has an obvious geometrical meaning:



$\lambda(w)$ is the orthogonal projection of $\lambda-\lambda(u_0)$ onto the subspace H_1.

Remark 2.3. Let us derive a "classical version" of the principle. Taking $v = \varphi \in C_0^\infty(\Omega)$ (i.e., an infinitely differentiable function with compact support in Ω) in the definition of $\Lambda_{f,g}$, we obtain

$$\lambda \in \Lambda_{f,g} \Rightarrow \int_\Omega \lambda_i \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega (f-\lambda_{n+1})\varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

We may define the operator $\operatorname{div}\lambda$ in the sense of distributions:

$$\int_\Omega \lambda_i \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \operatorname{div}\lambda\, dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

By comparison we conclude that

$$\lambda \in \Lambda_{f,g} \Rightarrow \operatorname{div}\lambda = \lambda_{n+1}-f \in L_2(\Omega). \tag{2.10}$$

Consequently, for $\lambda \in \Lambda_{f,g}$ we can also define a functional $\lambda_i\nu_i \in W^{-1/2,2}(\Gamma)$ by means of the relation

$$\langle \lambda_i\nu_i, \gamma v\rangle = \int_\Omega \left(\lambda_i \frac{\partial v}{\partial x_i}+v \operatorname{div}\lambda\right) dx, \tag{2.11}$$

where γv denotes the "trace" of a function $v \in W^{1,2}(\Omega)$ (see e.g. [1] for the concept of traces).

From (2.10) and (2.11) we obtain

$$(\lambda, \lambda(u_0))_H = \int_\Omega \left(\lambda_i\frac{\partial u_0}{\partial x_i}+\lambda_{n+1}u_0\right) dx$$

$$= \int_\Omega \left(\lambda_i\frac{\partial u_0}{\partial x_i}+u_0\operatorname{div}\lambda+u_0 f\right) dx = \langle\lambda_i\nu_i, \gamma u_0\rangle + \int_\Omega f u_0\, dx.$$

Next assume that the functional $\lambda_i\nu_i$ is represented by a function from $L_2(\Gamma)$, which is equal to g on Γ_g, i.e.,

$$\langle\lambda_i\nu_i, \gamma u_0\rangle = \int_{\Gamma_u} \lambda_i\nu_i\gamma u_0\, d\Gamma + \int_{\Gamma_g} g\gamma u_0\, d\Gamma.$$

Then a comparison with (2.7) yields that

$$\mathscr{S}(\lambda) = \tfrac{1}{2}\|\lambda\|_H^2 - \langle\lambda_i\nu_i, \gamma u_0\rangle - \int_\Omega f u_0\, dx + \tfrac{1}{2}\|\lambda(u_0)\|_H^2$$

$$= \tfrac{1}{2}\|\lambda\|_H^2 - \int_{\Gamma_u} \lambda_i\nu_i u_0\, d\Gamma + \mathscr{F}(f, g, u_0) = -\mathscr{S}_1(\lambda)+\mathscr{F}(f, g, u_0),$$

the term $\mathscr{F}$ being independent of λ.

The formulation

$$-\mathscr{S}_1(\lambda) = \min, \quad \lambda \in \Lambda_{f,g}$$

represents a more "classical" version of the principle. Our version, however, is more general, because no additional assumptions are included. ■

Having numerical methods in mind, we shall replace the affine hyperplane $\Lambda_{f,g}$ by the sum of a particular element $\lambda^0 \in \Lambda_{f,g}$ and the subspace H_2, i.e., we set

$$\Lambda_{f,g} = \lambda^0 + H_2 .$$

Then we write for $\chi \in H_2$

$$\mathscr{S}(\lambda) = \mathscr{S}(\lambda^0+\chi) = \tfrac{1}{2}\|\chi+\hat{\lambda}\|_H^2 = \tfrac{1}{2}\|\chi\|_H^2+(\chi, \hat{\lambda})_H+\tfrac{1}{2}\|\hat{\lambda}\|_H^2,$$

where

$$\hat{\lambda} = \lambda^0 - \lambda(u_0)\ .$$

is a fixed vector.

Hence the equivalent version of the principle follows:

$$\Phi_0(\chi) = \tfrac{1}{2}\|\chi\|_H^2 + (\chi, \hat{\lambda})_H = \min, \quad \chi \in H_2, \tag{2.12}$$

if and only if

$$\chi = \lambda(u) - \lambda^0.$$

As in the derivation of $\mathscr{S}_2$, we may exploit the particular form of (2.3). From (2.10) we know that

$$\chi \in H_2 = \Lambda_{0,0} \Rightarrow \operatorname{div}\chi = \chi_{n+1} \in L_2(\Omega)$$

and for $\chi_i \nu_i \in W^{-1/2,2}(\Gamma)$ we have, by virtue of (2.11),

$$\langle \chi_i \nu_i, \gamma v\rangle = \int_\Omega \left(\chi_i \frac{\partial v}{\partial x_i} + v\chi_{n+1}\right) dx = B(\chi, v) \quad \forall v \in W^{1,2}(\Omega),$$

$$\chi \in H_2 \Rightarrow \langle \chi_i \nu_i, \gamma v\rangle = 0 \quad \forall v \in V.$$

Write

$$\bar{\chi} = (\chi_1, \ldots, \chi_n)$$

and introduce the spaces of "reduced" vectors

$$Q = \{\bar{\chi} |\ \bar{\chi} \in [L_2(\Omega)]^n,\ \operatorname{div}\bar{\chi} \in L_2(\Omega)\}, \tag{2.13}$$

$$Q_0 = \{\bar{\chi} |\ \bar{\chi} \in Q, \langle \bar{\chi}_i \nu_i, \gamma v\rangle = 0\ \forall v \in V\}. \tag{2.14}$$

It is easy to prove that

$$\begin{cases} \bar{\chi} \in Q_0 \Rightarrow \chi = [\bar{\chi}_1, \ldots, \bar{\chi}_n, \operatorname{div}\bar{\chi}] \in H_2, \\ \chi \in H_2 \Rightarrow \chi = [\bar{\chi}_1, \ldots, \bar{\chi}_n, \operatorname{div}\bar{\chi}],\ \bar{\chi} \in Q_0. \end{cases} \tag{2.15}$$

Let us introduce

$$(\chi', \chi'')_Q = \int_\Omega (b_{ij}\chi_i'\chi_j'' + b_0 \operatorname{div}\chi' \operatorname{div}\chi'')\,dx$$

on $Q \times Q$. Then, obviously,

$$\|\chi\|_Q = (\chi, \chi)_Q^{1/2} = C\sum_{i=1}^n \|\chi_i\|_1 \quad \forall \chi \in [W^{1,2}(\Omega)]^n. \tag{2.16}$$

COROLLARY (Equivalent version of the principle of minimum complementary energy). *Let us define* $\bar{\lambda}^0 = \{\lambda_1^0, \ldots, \lambda_n^0\}$ *and*

$$\psi(\bar{\chi}) = \tfrac{1}{2}\|\bar{\chi}\|_Q^2 + (\bar{\chi}, \bar{\lambda}^0)_Q + \int_\Omega b_0 f \operatorname{div}\bar{\chi}\,dx - \langle \bar{\chi}_i \nu_i, u_0\rangle;$$

then

$$\psi(\bar{\chi}) = \min, \quad \bar{\chi} \in Q_0, \tag{2.17}$$

if and only if

$$\chi = [\bar{\chi}_1, \ldots, \bar{\chi}_n, \operatorname{div}\bar{\chi}] = \lambda(u) - \lambda^0,$$

where u is a solution of (1.2).

Proof. 1. We have

$$\chi = [\bar{\chi}, \operatorname{div}\bar{\chi}] \Rightarrow \Phi_0(\chi) = \psi(\bar{\chi}).$$

2. The difference $\lambda(u) - \lambda^0$ can be written in the form

$$\lambda(u) - \lambda^0 = [\bar{\chi}, \operatorname{div}\bar{\chi}], \quad \text{where} \quad \bar{\chi} \in Q_0.$$

In fact, from the definition of the weak solution u it follows that $\lambda(u) - \lambda^0 \in H_2$ (see the proof of Theorem 2.1) and $(2.15)_2$ holds.

Consequently, the problem (2.12) can be considered on the set of $\chi = [\bar{\chi}, \operatorname{div}\bar{\chi}]$, where $\bar{\chi} \in Q_0$. ∎

To define a Ritz–Galerkin procedure with finite elements, we shall assume that a family $\{V_h\}$ of subspaces (finite-dimensional) V_h exists such that for any h, $0 < h \leqslant 1$, the following conditions are satisfied:

(A1) $$V_h \subset W^{1,2}(\Omega),$$

(A2) an integer $\varkappa \geqslant 2$ and a constant C exist, independent of h and such that

$$\forall v \in W^{\varkappa,2}(\Omega)\ \exists v_h \in V_h, \quad \|v_h - v\|_1 \leqslant Ch^{\varkappa-1}\|v\|_\varkappa.$$

Writing

$$V(h) = [V_h]^n,$$

we have

$$V(h) \subset [W^{1,2}(\Omega)]^n \subset Q.$$

Remark 2.4. The well-known spaces of piecewise polynomial functions defined on a simplicial partition of a polyhedral domain Ω satisfy (A1) and (A2). For example, the linear polynomials correspond to $\varkappa = 2$, and cubic to $\varkappa = 4$.

Let

$$V_0(h) = Q_0 \cap V(h) = \{\chi |\ \chi \in [V_h]^n,\ \chi_i \nu_i = 0 \text{ on } \Gamma_g\}.$$

We say that $\chi^h \in V_0(h)$ is a *finite-element approximation* of the problem (2.17) if

$$\psi(\chi^h) = \min \psi(\chi), \quad \chi \in V_0(h). \tag{2.18}$$

THEOREM 2.2. *Let* $\Gamma = \Gamma_u$ *and let the boundary* Γ *be sufficiently smooth. Setting* $\lambda^0 = \{0, \ldots, 0, f\}$, $\bar{\lambda}^h = \chi^h$, $\bar{\lambda}(u) = \{\lambda_1(u), \ldots, \lambda_n(u)\}$, *we have*

$$\lim_{h \to 0} \|\bar{\lambda}^h - \bar{\lambda}(u)\|_Q = 0.$$

If $\bar{\lambda}(u) \in [W^{\varkappa,2}(\Omega)]^n$, *then*

$$\|\bar{\lambda}^h - \bar{\lambda}(u)\|_Q = Ch^{\varkappa-1}\sum_{i=1}^n \|\bar{\lambda}(u)_i\|_\varkappa.$$

Proof. The crucial point consists in the following lemma (Aubin, Burchard [2]): *if the boundary is sufficiently smooth, then* $[C^\infty(\bar{\Omega})]^n$ *is dense in* Q.

As $\Gamma_g = \emptyset$, we may set $Q_0 = Q$, $V_0(h) = V(h)$, $V = W_0^{1,2}(\Omega)$. From the above-mentioned density we deduce

$$\forall \varepsilon > 0\ \exists \varphi \in [C^\infty(\bar{\Omega})]^n, \quad \|\bar{\lambda}(u)-\varphi\|_Q < \varepsilon/2.$$

Assumption (A2) yields

$$\forall h\ \exists \varphi^h \in V(h), \quad \sum_{i=1}^n \|\varphi_i^h - \varphi_i\|_1 \leqslant Ch^{\varkappa-1}\sum_{i=1}^n \|\varphi_i\|_\varkappa,$$

because $\varphi \in [W^{\varkappa,2}(\Omega)]^n$.

Consequently, using also (2.16), we have

(2.19) $\quad \forall \varepsilon > 0\ \exists h_0,\ \forall h < h_0\ \exists \varphi^h \in V(h),$

$$\|\bar{\lambda}(u)-\varphi^h\|_Q \leqslant \|\bar{\lambda}(u)-\varphi\|_Q + \|\varphi-\varphi^h\|_Q < \varepsilon/2 + C_1 h^{\varkappa-1}\sum_{i=1}^n \|\varphi_i\|_\varkappa < \varepsilon.$$

Now (2.17) results in

$$(\bar{\lambda}(u), \mu)_Q = l(\mu) \quad \forall \mu \in Q \supset V(h),$$

and similarly (2.18) in

$$(\bar{\lambda}^h, \mu)_Q = l(\mu) \quad \forall \mu \in V(h),$$

where

$$l(\mu) = -\int_\Omega b_0 f \operatorname{div}\mu\, dx + \langle \mu_i \nu_i, u_0\rangle.$$

By subtraction we obtain

$$(\bar{\lambda}(u)-\bar{\lambda}^h, \mu)_Q = 0 \quad \forall \mu \in V(h),$$

i.e., $\bar{\lambda}^h$ is the orthogonal projection of $\bar{\lambda}(u)$ onto $V(h)$. Therefore

(2.20) $$\|\bar{\lambda}(u)-\bar{\lambda}^h\|_Q \leqslant \|\bar{\lambda}(u)-\mu\|_Q \quad \forall \mu \in V(h).$$

Finally, from (2.19) and (2.20) it follows that

$$\|\bar{\lambda}(u)-\bar{\lambda}^h\|_Q \leqslant \|\bar{\lambda}(u)-\varphi^h\|_Q < \varepsilon.$$

If $\bar{\lambda}(u) \in [W^{\varkappa,2}(\Omega)]^n$, we need not the density in Q and the assertion of the theorem is an easy consequence of (2.20), (2.16) and (A2). ■

Now let us consider the case $\Gamma_g \neq \emptyset$ and restrict ourselves for brevity to plane problems on a polygonal domain $\Omega \subset \boldsymbol{R}^2$. Assume that the subspaces V_h are constructed by means of triangulations $\mathscr{T}_h$ of Ω, which satisfy the following requirement:

(B1) If a part of Γ_g belongs to a side of a triangle $K \in \mathscr{T}_h$, then $\bar{\Gamma}_g$ covers the whole side.

Let us find $\lambda^0 \in \Lambda_{f,g}$. (Choosing a $\bar{\lambda}^0 \in Q$ such that $\bar{\lambda}_i^0 \nu_i = g$ on Γ_g, we set $\lambda_{n+1}^0 = \operatorname{div}\bar{\lambda}^0 + f$.)

Define

$$\mathscr{V} = Q_0 \cap [W^{\varkappa,2}(\Omega)]^2,$$

and denote by $\bar{\mathscr{V}}^Q$ the closure of $\mathscr{V}$ in Q.

THEOREM 2.3. *Let the subspaces V_h be constructed by means of Lagrange or Hermite interpolation on triangulations $\mathscr{T}_h$, satisfying* (B1).

If $\bar{\lambda}(u)-\bar{\lambda}^0 \in \bar{\mathscr{V}}^Q$, then the finite-element approximations $\bar{\lambda}^0+\chi^h = \bar{\lambda}^h$ converge in Q to $\bar{\lambda}(u)$, i.e.,

$$\lim_{h\to 0}\|\bar{\lambda}(u)-\bar{\lambda}^h\|_Q = 0.$$

If $\lambda(u)-\lambda^0 \in \mathscr{V}$, then

$$\|\bar{\lambda}(u)-\bar{\lambda}^h\|_Q \leqslant Ch^{\varkappa-1}\sum_{i=1}^2 \|\bar{\lambda}(u)_i - \bar{\lambda}_i^0\|_\varkappa.$$

Proof. The fact that $\mathscr{V}$ is dense in $\bar{\mathscr{V}}^Q$ yields that

(2.21) $$\forall \varepsilon > 0\ \exists \varphi \in \mathscr{V}, \quad \|\bar{\chi}^0-\varphi\|_Q \leqslant \varepsilon/2, \quad \bar{\chi}^0 = \bar{\lambda}(u)-\bar{\lambda}^0.$$

Using the Lagrange or Hermite interpolation on triangles we construct a linear mapping (see e.g. [5], [6])

$$r_h \in \mathscr{L}(W^{\varkappa,2}(\Omega), V_h)$$

such that

(2.22) $$\|\varphi_i - r_h\varphi_i\|_1 \leqslant Ch^{\varkappa-1}\|\varphi_i\|_\varkappa \quad (i = 1, 2),$$

where C is independent of h and φ_i.

Moreover, we have

$$\varphi \in \mathscr{V} \Rightarrow r_h\varphi \equiv (r_h\varphi_1, r_h\varphi_2) \in Q_0.$$

In fact, e.g. for the Lagrange interpolation, we have

$$\varphi \in \mathscr{V} \Rightarrow \varphi_i\nu_i = 0 \quad \text{on } \Gamma_g,$$

because $\mathscr{V} \subset [W^{\varkappa,2}(\Omega)]^2 \subset [C(\bar{\Omega})]^2$ holds for $\varkappa \geqslant 2$.

At the nodal points of $\bar{\Gamma}_g$ we have

$$0 = \varphi_i\nu_i = (r_h\varphi_i)\nu_i;$$

consequently

$$(r_h\varphi_i)\nu_i = 0 \text{ on } \Gamma_g \Rightarrow r_h\varphi \in Q_0.$$

An analogous approach is applicable in the case of the Hermite interpolation. Thus

$$r_h\varphi \in Q_0 \cap [V_h]^2 = V_0(h).$$

By virtue of (2.16) and (2.21), (2.22),

$$\|\bar{\chi}^0 - r_h\varphi\|_Q \leqslant \|\bar{\chi}^0-\varphi\|_Q + \|\varphi - r_h\varphi\|_Q < \varepsilon$$

holds for sufficiently small h.

As χ^h is the orthogonal projection of $\bar{\chi}^0$ onto $V_0(h)$ in Q (cf. (2.20)), it holds

$$\|\bar{\lambda}(u)-\bar{\lambda}^h\|_Q = \|\bar{\chi}^0-\chi^h\|_Q \leqslant \|\bar{\chi}^0 - r_h\varphi\|_Q < \varepsilon.$$

If $\bar{\chi}^0 \in \mathscr{V}$, we may set $\varphi = \bar{\chi}^0$ in the above proof.

Remark 2.5. A comparison of numerical efficiency of both the primal method (with compatible models) and the dual method (with equilibrium models) can be found in [7]. It results in favour of the dual method.

Remark 2.6. In parabolic mixed problems, the role of the "absolute" term is played by the time-derivative $\partial u/\partial t$.

A conjugate variational formulation can also be established and utilized, yielding efficient finite element procedures for computing the cogradient vector, see [8], [9].

3. Equation without an absolute term

Let us consider equation (1.1), where $a_0 \equiv 0$ and let $\Gamma_v = \emptyset$; $\Gamma_u \neq \emptyset$. Using the Friedrichs transform (without the variable η_{n+1}), we again derive $(2.2)_1$, (2.4), (2.5), (2.6), but instead of (2.3) we obtain

$$\operatorname{div}\lambda + f = 0. \tag{2.3'}$$

We are led to the variational problem

$$\mathcal{S}_3(\lambda) = \max, \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$$

where

$$\mathcal{S}_3(\lambda) = \mathcal{S}_1(\lambda_1, \ldots, \lambda_n, 0)$$

and Λ consist of vectors which satisfy (2.3)′ and (2.4).

To analyze the latter problem, we introduce (see [4]) $H = [L_2(\Omega)]^n$,

$$(\lambda', \lambda'')_H = \int_\Omega b_{ij}\lambda_i'\lambda_j''\,dx,$$

so that

$$\|\lambda\|_H^2 = (\lambda, \lambda)_H \geqslant c_2 \sum_{i=1}^n \|\lambda_i\|_0^2$$

and H is a Hilbert space.

Then we introduce (cf. Definition 2.1)

$$B(\lambda, v) = \int_\Omega \lambda_i \frac{\partial v}{\partial x_i}\,dx,$$

$$H_1 = \left\{\lambda \in H \mid \exists v \in V,\ \lambda_i = a_{ij}\frac{\partial v}{\partial x_j}\ (i = 1, \ldots, n) \equiv \lambda = \lambda(v)\right\},$$

$$H_2 = \{\lambda \in H \mid B(\lambda, v) = 0\ \forall v \in V\},$$

$$\Lambda_{f,g} = \{\lambda \in H \mid B(\lambda, v) = F(v)\ \forall v \in V\}.$$

THEOREM 3.1 (Principle of minimum complementary energy). *Define*

$$\mathcal{S}(\lambda) = \tfrac{1}{2}\|\lambda - \lambda(u_0)\|_H^2.$$

Then

$$\mathcal{S}(\lambda) = \min, \qquad \lambda \in \Lambda_{f,g},$$

if and only if $\lambda = \lambda(u)$, *where* u *is a solution of* (1.2).

Proof is similar to that of Theorem 2.1.

Introducing a particular fixed $\lambda^0 \in \Lambda_{f,g}$, we derive an equivalent version of the principle:

$$\Phi_0(\chi) = \tfrac{1}{2}\|\chi\|_H^2 + (\chi, \hat{\lambda})_H = \min, \qquad \chi \in H_2, \qquad \text{where} \quad \hat{\lambda} = \lambda^0 - \lambda(u_0), \tag{3.1}$$

if and only if $\chi = \lambda(u) - \lambda^0$.

Let a family $\{S_h\}$, $0 < h \leqslant 1$, of subspaces $S_h \subset H_2$ be given. We say that $\chi^h \in S_h$ is a *finite-element approximation* to the problem (3.1), if

$$\Phi_0(\chi^h) = \min \Phi_0(\chi), \qquad \chi \in S_h. \tag{3.2}$$

In what follows, we show a possible *construction of* S_h, using *piecewise linear functions*.

For simplicity, we consider plane polygonal domain $\Omega \subset \boldsymbol{R}^2$. (For an extension to $\boldsymbol{R}^n$, see [10].) First we introduce a projection mapping on a single triangle $K \subset \Omega$. Write

$$W(K) = [W^{1,2}(K)]^2, \qquad C(K) = [C(K)]^2,$$

and denote by $P_1(M)$ the space of all linear polynomials defined on the set M. Let $a_1, a_2, a_3, a_4 = a_1$ be the vertices of K, $\nu^{(i)}$ the unit outward normal to the side $\overline{a_i a_{i+1}}$.

For $\lambda \in W(K)$ we define the "outward flux"

$$T_i\lambda = \lambda_j \nu_j^{(i)}|_{\overline{a_i a_{i+1}}}.$$

LEMMA 3.1. *Let* $\alpha_i, \beta_i \in \boldsymbol{R}^1$ $(i = 1, 2, 3)$ *be given. Then precisely one* $\lambda \in [P_1(K)]^2$ *exists such that*

$$T_i\lambda(a_i) = \alpha_i, \qquad T_i\lambda(a_{i+1}) = \beta_i \qquad (i = 1, 2, 3). \blacksquare$$

Henceforth we shall use the notation

$$\int_{\overline{a_i a_{i+1}}} uv\,ds = [u, v]_i$$

and the basic linear functions $\omega_1^{(i)}, \omega_2^{(i)} \in P_1(\overline{a_i a_{i+1}})$ of the side $\overline{a_i a_{i+1}}$, for which

$$\omega_1^{(i)}(a_i) = 1, \qquad \omega_1^{(i)}(a_{i+1}) = 0,$$

$$\omega_2^{(i)}(a_i) = 0, \qquad \omega_2^{(i)}(a_{i+1}) = 1.$$

Let us project each $T_i\lambda$ onto $P_1(\overline{a_i a_{i+1}})$ in $L_2(\overline{a_i a_{i+1}})$ and find the corresponding vector $\lambda \in [P_1(K)]^2$. Thus we obtain a maping Π; more precisely, we have

THEOREM 3.2. *Let* $\lambda \in W(K)$. *Then the equations*

$$[T_i\lambda, \omega_k^{(i)}]_i = \alpha_i[\omega_1^{(i)}, \omega_k^{(i)}]_i + \beta_i[\omega_2^{(i)}, \omega_k^{(i)}]_i \qquad (k = 1, 2), \tag{$*$}$$

$$(**)\qquad T_i\Pi\lambda(a_i) = \alpha_i,\qquad T_i\Pi\lambda(a_{i+1}) = \beta_i,$$

with $i = 1, 2, 3$, *define a mapping*

$$\Pi \in \mathscr{L}(W(K); [P_1(K)]^2)\cap\mathscr{L}(C(K); [P_1(K)]^2).$$

Moreover,

$$\|\Pi\lambda\|_{C(K)} \leqslant \frac{6\sqrt{2}}{\sin\alpha}\|\lambda\|_{C(K)}\qquad \forall\lambda\in C(K),$$

where α is the minimal angle of K.

For the proofs of this and the following results, see [4]. ■.

Let us define

$$\mathscr{M}(K) = \{\lambda\in[P_1(K)]^2, \operatorname{div}\lambda = 0\}.$$

It is easy to find that $\dim\mathscr{M}(K) = 5$,

$$\mathscr{M}(K) = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x_2\\0\end{pmatrix}, \begin{pmatrix}0\\x_1\end{pmatrix}, \begin{pmatrix}x_1\\-x_2\end{pmatrix}\right\}.$$

LEMMA 3.2. *Let* $\lambda\in[P_1(K)]^2$. *Then*

$$\lambda\in\mathscr{M}(K)\Leftrightarrow\sum_{i=1}^{3}(\alpha_i+\beta_i)l_i = 0,$$

where $\alpha_i = T_i\lambda(a_i)$, $\beta_i = T_i\lambda(a_{i+1})$, l_i *is the length of* $\overline{a_ia_{i+1}}$.

THEOREM 3.3. *Let Π be defined by means of* (*), (**). *Define*

$$U(K) = \{\lambda\in W(K), \operatorname{div}\lambda = 0\}.$$

Then

$$\Pi\in\mathscr{L}(U(K); \mathscr{M}(K)),$$
$$\Pi\lambda = \lambda\qquad\forall\lambda\in[P_1(K)]^2.$$

If $\lambda\in[C^2(K)]^2$, *then*

$$\|\lambda-\Pi\lambda\|_{C(K)} \leqslant 4\left(1+\frac{6\sqrt{2}}{\sin\alpha}\right)h^2\|\lambda\|_{[C^2(K)]^2},$$

where $h = \operatorname{diam}K$ *and α is the minimal angle of K.* ■

Let us consider a triangulation $\mathscr{T}_h$ of Ω. Write

$$h = \max_{K\in\mathscr{T}_h}\operatorname{diam}K,$$

and let Π_K be the mapping defined on $K\in\mathscr{T}_h$ by (*), (**). In order to guarantee the continuity of fluxes across each common side of two adjacent triangles $K, K'\in\mathscr{T}_h$, we introduce the following

CONDITION (R):

$$T_{i,K}\lambda+T_{i,K'}\lambda = 0$$

holds on each common side of any two adjacent triangles of $\mathscr{T}_h$.

Here $T_{i,K}\lambda = \lambda_j\nu_j^{(i)}(K)$ and $T_{i,K'}\lambda = \lambda_j\nu_j^{(i)}(K')$, $\nu^{(i)}(K) = -\nu^{(i)}(K')$.

Define

$$U(\Omega) = \{\lambda\in[W^{1,2}(\Omega)]^2, \operatorname{div}\lambda = 0\},$$
$$\mathscr{N}_h(\Omega) = \{\lambda|\ \lambda|_K\in\mathscr{M}(K)\ \forall K\in\mathscr{T}_h,\ \lambda \text{ satisfies condition (R)}\}.$$

Remark 3.1. Any $\lambda\in\mathscr{N}_h(\Omega)$ satisfies the equation $\operatorname{div}\lambda = 0$ in Ω in the sense of distributions.

For $\lambda\in U(\Omega)$ we define a mapping r_h as follows:

$$(r_h\lambda)|_K = \Pi_K\lambda\qquad\forall K\in\mathscr{T}_h.$$

THEOREM 3.4. *Let $\{\mathscr{T}_h\}$ be a regular* (¹), *family of triangulations. Then*

$$r_h\in\mathscr{L}(U(\Omega); \mathscr{N}_h(\Omega)),$$
$$\|\lambda-r_h\lambda\|_{[L_2(\Omega)]^2} \leqslant C_0h^2\|\lambda\|_{[C^2(\bar\Omega)]^2}\qquad\forall\lambda\in[C^2(\bar\Omega)]^2,$$

where C_0 is independent of h and λ. ■

Let now $\{\mathscr{T}_h\}$ be a regular family of triangulations satisfying also the requirement (B1). We define

$$S_h = \mathscr{N}_h(\Omega)\cap H_2 = \{\lambda\in\mathscr{N}_h(\Omega)|\ \lambda_j\nu_j = 0 \text{ on } \Gamma_g\}$$

and denote by $\chi^0\in H_2$ the element for which

$$\Phi_0(\chi^0) = \min\Phi_0(\chi),\qquad \chi\in H_2.$$

THEOREM 3.5. *Let* $\chi^0\in[C^2(\bar\Omega)]^2$. *Then*

$$\|\chi^0-\chi^h\|_H \leqslant Ch^2\|\chi^0\|_{[C^2(\bar\Omega)]^2}$$

(*where χ^h is defined in* (3.2)) *holds for any regular family of triangulations.*

COROLLARY. *Let the suppositions of Theorem 3.5 be satisfied. Setting*

$$\lambda^0+\chi^0 = \lambda(u),\qquad \lambda^0+\chi^h = \lambda^h,$$

we obtain

$$\|\lambda(u)-\lambda^h\|_H = O(h^2).$$

Remark 3.2. If $\lambda^0\in\Lambda_{f,g}$ is not available, we may replace it by an approximate λ_g^0 which corresponds to a piecewise linear interpolate of g on Γ_g. The same error estimate can be derived.

Remark 3.3. *A posteriori error estimates.* Suppose that we have attacked the problem from two sides:

(i) by the primary method, using a compatible finite element model, which yields an approximation

$$u_{h^*}\in u_0+v_{h^*},\qquad v_{h^*}\in V_{h^*}\subset V;$$

(ii) by the dual method, using the equilibrium model described above, which yields

$$\lambda^h = \lambda^0+\chi^h,\qquad \chi^h\in S_h,\qquad \lambda^0\in\Lambda_{f,g}.$$

(¹) The family of triangulations is regular if a positive $\alpha_0 > 0$ exists such that $\inf_h\min_{K\in\mathscr{T}_h}\alpha \geqslant \alpha_0$.

THEOREM 3.6. *The following estimates hold*:

$$C\|u_{h^*}-u\|_1 \leqslant \|\lambda(u_{h^*})-\lambda(u)\|_H \leqslant \|\lambda(u_{h^*})-\lambda^h\|_H,$$

$$C_1\|\lambda^h-\lambda(u)\|_{[L_2(\Omega)]^2} \leqslant \|\lambda^h-\lambda(u)\|_H \leqslant \|\lambda(u_{h^*})-\lambda^h\|_H.$$

The proof is an immediate consequence of the equation

$$\|\lambda(u_{h^*})-\lambda^h\|_H^2 = \|\lambda(u_{h^*})-\lambda(u)\|_H^2+\|\lambda(u)-\lambda^h\|_H^2.$$

The case $a_0 > 0$ can be treated in a similar way (cf. also [2]).

4. A mixed finite element method close to the equilibrium model

For the Dirichlet problem with constant coefficients, we present a new variational formulation, by means of which a converging mixed model can be established.

Consider the equation

$$-a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \text{in } \Omega \subset R^n, \tag{4.1}$$

$$u = 0 \quad \text{on } \Gamma,$$

where $a_{ij} = a_{ji} = \text{const}$, $f \in L_2(\Omega)$ and the matrix $[a_{ij}]$ is positive definite.

Applying the Friedrichs transform to the problem $\mathscr{L}(u) = \min$ with

$$\eta_i = \frac{\partial u}{\partial x_i}+\alpha_i u, \quad i = 1, 2, \ldots, n, \quad \eta_{n+1} = u,$$

where $\alpha \in R^n$ is an arbitrary nonzero constant vector, we derive a new variational formulation of the problem (4.1).

DEFINITION 4.1. Let us introduce (cf. (2.13))

$$Q = \{\lambda |\ \lambda \in [L_2(\Omega)]^n,\ \operatorname{div}\lambda \in L_2(\Omega)\},$$

$$\|\lambda\|_Q = \sum_{j=1}^{n} \|\lambda_j\|_0+\|\operatorname{div}\lambda\|_0,$$

$$B(\lambda, \mu) = (b_{ij}\lambda_i, \mu_j)-\gamma^{-1}(\operatorname{div}\lambda-\alpha_j\lambda_j, \operatorname{div}\mu-\alpha_i\mu_i),$$

where $[b] = [a]^{-1}$, $\gamma = a_{ij}\alpha_i\alpha_j$,

$$(u, v) = \int_\Omega uv\,dx.$$

THEOREM 4.1. *The variational problem of finding $\lambda \in Q$ such that*

$$B(\lambda, \mu) = \gamma^{-1}(f, \operatorname{div}\mu-\alpha_i\mu_i) \quad \forall \mu \in Q \tag{4.2}$$

has a unique solution λ^*, where

$$\lambda_i^* = a_{ij}\left(\frac{\partial u}{\partial x_j}+\alpha_j u\right), \quad i = 1, 2, \ldots, n,$$

and u is the (weak) solution of the problem (4.1).

Moreover,

$$u = -\gamma^{-1}(\operatorname{div}\lambda^*-\alpha_j\lambda_j^*+f),$$

$$\|\lambda^*\|_Q \leqslant C\|f\|_0.$$

Let us take for simplicity $a_{ij} = \delta_{ij}$, i.e., the equation $-\Delta u = f$. Sometimes, one component of the gradient $\frac{\partial u}{\partial x_i}$, say $\frac{\partial u}{\partial x_n}$, is of less interest than $\frac{\partial u}{\partial x_i}$, $i \leqslant n-1$. (This may be the case if e.g. the domain Ω is "thin" in the direction of x_n.) Then we can set

$$\alpha_i = 0 \quad \text{for} \quad i \leqslant n-1, \quad \alpha_n = \alpha_0 h^{-1-\varepsilon},$$

where $0 < h \leqslant 1$, $\alpha_0 > 0$, $\varepsilon > 0$ are some parameters, and transform the last component of the gradient as follows:

$$\lambda_n = \alpha_n\bar{\lambda}_n.$$

Introducing

$$\bar{\lambda} = (\lambda_1, \ldots, \bar{\lambda}_n),$$

$$\bar{B}(\lambda_1, \ldots, \bar{\lambda}_n; \mu_1, \ldots, \bar{\mu}_n) = B(\lambda; \mu),$$

$$\tilde{B}(\bar{\lambda}; \bar{\mu}) = \bar{B}(\lambda_1, \ldots, \bar{\lambda}_n; \mu_1, \ldots, \bar{\mu}_n),$$

we may replace the problem (4.2) by an equivalent one:

$$\tilde{B}(\bar{\lambda}; \bar{\mu}) = \alpha_n^{-2}\left(f, \sum_{j=1}^{n-1}\frac{\partial\mu_j}{\partial x_j}-\alpha_n\frac{\partial\bar{\mu}_n}{\partial x_n}+\alpha_n^2\bar{\mu}_n\right), \tag{4.3}$$

$$\forall \bar{\mu} \in Q_{\bar{0}} = \left\{\bar{\mu} \in [L_2(\Omega)]^n, \sum_{j=1}^{n-1}\frac{\partial\mu_j}{\partial x_j}-\alpha_n\frac{\partial\bar{\mu}_n}{\partial x_n} \in L_2(\Omega)\right\},$$

$$\bar{\lambda} \in Q_0 = \left\{\bar{\lambda} \in [L_2(\Omega)]^n, \sum_{j=1}^{n-1}\frac{\partial\lambda_j}{\partial x_j}+\alpha_n\frac{\partial\bar{\lambda}_n}{\partial x_n} \in L_2(\Omega)\right\}.$$

Suppose we have two families of subspaces:

$$\{V_h\} \text{ and } \{V_{h_n}\} \quad \text{with} \quad 0 < h \leqslant 1,\ 0 < h_n \leqslant 1,$$

satisfying the following requirements:

(i) $V_h \subset W^{1,2}(\Omega)$, $V_{h_n} \subset W_0^{1,2}(\Omega)$ $\forall h, h_n$,

(ii) $\exists$ integers $\varkappa \geqslant 2$, $\varkappa_n \geqslant 2$ and $C = \text{const}$ such that $\forall v \in W^{\varkappa,2}(\Omega) \exists v_h \in V_h$:

$$\|v-v_h\|_i \leqslant Ch^{\varkappa-i}\|v\|_\varkappa, \quad i = 0, 1,$$

$\forall w \in W^{\varkappa_n,2}(\Omega)\cap W_0^{1,2}(\Omega) \exists w_h \in V_{h_n}$:

$$\|w-w_h\|_i \leqslant Ch_n^{\varkappa_n-i}\|w\|_{\varkappa_n}, \quad i = 0, 1,$$

(iii) $\exists C_0 = \text{const}$ such that for sufficiently small h

$$\|\chi\|_1 \leqslant C_0 h^{-1}\|\chi\|_0 \quad \forall \chi \in V_h.$$

Denote by $V(h, h_n)$ the space

$$V(h, h_n) = [V_h]^{n-1} \times V_{h_n}.$$

We shall say that $\bar{\lambda}^h \in V(h, h_n)$ is a *finite-element approximation of the problem* (4.3), if

$$\tilde{B}(\bar{\lambda}^h, \bar{\mu}) = \alpha_n^{-2}\left(f, \sum_{j=1}^{n-1} \frac{\partial \mu_j}{\partial x_j} - \alpha_n \frac{\partial \bar{\mu}_n}{\partial x_n} + \alpha_n^2 \bar{\mu}_n\right) \quad \forall \bar{\mu} \in V(h, h_n).$$

THEOREM 4.2. *Let the solution u belong to $W^{m,2}(\Omega)$, where $m \geqslant \max(\varkappa+1, \varkappa_n)$. Then, for sufficiently small h, the finite element approximation is determined uniquely and it holds*

$$\sum_{j=1}^{n-1} \left\| \frac{\partial u}{\partial x_j} - \lambda_j^h \right\|_0 + \left\| \frac{\partial u}{\partial x_n} - \frac{\partial \bar{\lambda}_n^h}{\partial x_n} \right\|_0 + \|u - \bar{\lambda}_n^h\|_0 \leqslant C(h^{\varkappa-1} + h^{\varepsilon} + h^{-1} h_n^{\varkappa_n} + h_n^{\varkappa_n - 1})\|u\|_m.$$

The proof can be found in [11].

EXAMPLE. Suppose that Ω is a polygonal domain in R^2 and $u \in W^{3,2}(\Omega)$. Setting $\varepsilon = 1$, $h_2 = Ch$ and using the piecewise linear polynomials on the triangulations $\mathcal{T}_h$ and $\mathcal{T}_{h_n}$, respectively, we obtain $\varkappa = \varkappa_n = 2$ and from Theorem 4.2 we get

$$\left\| \frac{\partial u}{\partial x_1} - \lambda_1^h \right\|_0 + \left\| \frac{\partial u}{\partial x_2} - \frac{\partial \bar{\lambda}_2^h}{\partial x_2} \right\|_0 + \|u - \bar{\lambda}_2^h\|_0 \leqslant Ch\|u\|_3.$$

Remark 4.1. In the case of a smooth boundary the curved elements along the boundary can be used for V_h, V_{h_n} (cf. [13], [14]). For the proof of convergence see [12].

The method can be extended to elliptic systems such as those of linear plane elastostatics. The corresponding mixed model is proposed and analyzed in [15].

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