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THREE-LEVEL DISCRETE TIME GALERKIN APPROXIMATIONS FOR THE NON-STATIONARY NAVIER-STOKES EQUATION

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Let Ω be a bounded domain in R^n $(2 \leqslant n \leqslant 4)$ with a boundary Γ. We shall consider several discrete time Galerkin methods for approximating solutions of the non-stationary Navier–Stokes equation of the form

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^{n} u_i D_i u = f - \nabla p, \quad \nu > 0, \tag{1}$$
$$\operatorname{div} u = 0$$

with the following boundary and initial conditions:

$$u|_{\Gamma} = 0, \quad u|_{t=0} = u_0,$$

where $u\colon \bar{Q} \to R^n$, $p\colon Q \to R$, $Q = \Omega \times (0, T)$, are unknown functions and Δ, ∇ denote the Laplace operator and the gradient, respectively.

The three-level discrete time Galerkin method, which results from applying the Galerkin procedure in the space variables and using three-level finite difference approximations in the time variable, was used in [2] to obtain an approximating solutions of nonlinear parabolic equations. In this paper we shall prove a theorem on the convergence of approximating solutions, obtained by two variants of the three-level discrete time Galerkin procedure, to the exact solution of the Navier–Stokes equation. The method of proof is very similar to that used by R. Temam in [6] in the proof of the theorem on the convergence of the two-level discrete time Galerkin procedure (cf. [3], [5]), which is in fact a simple modification of the method of proving the existence theorem for the Navier–Stokes equation (see [4]).

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We define the spaces

$$V = \{v\colon v \in (H_0^1(\Omega))^n, \operatorname{div} v = 0\}$$

and

$$H = \text{the closure of } V \text{ in } (L^2(\Omega))^n,$$

which are Hilbert spaces with the inner products

$$((f, g)) = \sum_{i,j=1}^{n} (D_j f_i, D_j g_i)_{L^2(\Omega)}, \quad (f, g) = \sum_{i=1}^{n} (f_i, g_i)_{L^2(\Omega)},$$

and with the norms

$$\|f\| = ((f,f))^{1/2}, \quad |f| = (f,f)^{1/2},$$

respectively, for $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$.

Define $L^2(0, T; X)$ and $L^\infty(0, T; Y)$ as the spaces of functions f: $[0, T] \to X$, g: $[0, T] \to Y$ such that the norms

$$\|f\|_{L^2(0,T;X)} = \Big(\int_0^T \|f(t)\|_X^2 dt\Big)^{1/2},$$

$$\|g\|_{L^\infty(0,T;Y)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|g(t)\|_Y,$$

are finite, where X and Y are Hilbert spaces and T is a fixed positive real number.

Now we formulate the problem which will be considered below. It can be posed as follows: for given

(2) $$f \in L^2(0, T; H),$$

(3) $$u_0 \in H,$$

find

(4) $$u \in L^2(0, T; V) \cap L^\infty(0, T; H)$$

such that

(5) $$\forall v \in V \quad (u', v) + \nu((u, v)) + b(u, u, v) = (f, v), \quad \nu > 0,$$

(6) $$u(0) = u_0,$$

where

(7) $$b(u, v, w) = \sum_{i,j=1}^{n} \int_\Omega u_i (D_i v_j) w_j dx \quad \text{for} \quad u, v, w \in V.$$

The problem (2)–(7) is equivalent to the following one: for given f, u_0, which satisfy (2), (3), find a function u on Q which satisfies (4) and (6) and a distribution p on Q such that equation (1) is fulfilled in the weak sense (see [4]).

3

Consider a family $\{V_h\}_{h \in G}$ of finite-dimensional subspaces of V, which satisfies the following conditions:

(8) $$\overline{\bigcup_{h \in G} V_h} = V,$$

(9) $$\forall h_1, h_2 \in G \quad h_1 > h_2 \Rightarrow V_{h_1} \subset V_{h_2},$$

(10) $$\forall h \in G \ \forall v_h \in V_h \forall 1 \leqslant i \leqslant n \quad D_i v_h \in L^n(\Omega),$$

where the set $G \subset (0, \infty)$ of parameters h has an accumulation point at 0. (An example of such a family is described in [6].)

It is obvious that

(11) $$\exists C > 0 \forall h \in G \forall u_h \in V_h \quad |u_h| \leqslant C\|u_h\|,$$

(12) $$\forall h \in G \ \exists S(h) > 0 \forall u_h \in V_h \quad \|u_h\| \leqslant S(h)|u_h|,$$

(13) $$\forall h \in G \ \exists R(h) > 0 \forall u_h, v_h \in V_h$$
$$|b(u_h, u_h, v_h)| \leqslant R(h)|u_h| \ \|u_h\| \ |v_h|.$$

Condition (11) follows from the definition of the norm $\|\cdot\|$, by Poincaré's inequality (see for example [1]), (12) is satisfied in view of the fact that every two norms in a finite dimensional linear space are equivalent, and (13) follows from (11), (12) and the continuity of the functional b on $V \times V \times V$.

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R. Temam in [6] discussed the two-level discrete time Galerkin method of the form

(14) $\forall v_h \in V_h$

$$\frac{1}{k}(u_{hk}^m - u_{hk}^{m-1}, v_h) + \nu((u_{hk}^m, v_h)) + b(u_{hk}^{m-1}, u_{hk}^m, v_h) = (f^m, v_h),$$

for $m = 1, \ldots, N$, $k = T/N$,

$$f^m = \frac{1}{k} \int_{(m-1)k}^{mk} f(t)dt, \quad m = 1, \ldots, N,$$

and for given u_{hk}^0.

The purpose of this paper is to give a theorem on the convergence of the following two variants of the three-level discrete time Galerkin scheme:

(15) $\forall v_h \in V_h$

$$\frac{1}{2k}(u_{hk}^{m+1} - u_{hk}^{m-1}, v_h) + \nu((u_{hk}^{m+1}, v_h)) + b(u_{hk}^m, u_{hk}^{m+1}, v_h) = (f^m, v_h),$$

(16) $\forall v_h \in V_h$

$$\frac{1}{2k}(u_{hk}^{m+1} - u_{hk}^{m-1}, v_h) + \nu((u_{hk}^{m\theta}, v_h)) + b(u_{hk}^m, u_{hk}^{m\theta}, v_h) = (f^m, v_h),$$

$$u_{hk}^{m\theta} = \theta u_{hk}^{m+1} + (1-2\theta)u_{hk}^m + \theta u_{hk}^{m-1},$$

for $m = 1, \ldots, N-1$, $k = T/N$,

(17) $$f^m = \frac{1}{2k} \int_{(m-1)k}^{(m+1)k} f(t)dt, \quad m = 1, \ldots, N-1$$

(N is a given natural even number), and for u_{hk}^0, u_{hk}^1 defined as follows:

(18) u_{hk}^0 is the orthogonal $(L^2(\Omega))^n$ projection of u_0 onto V_h,

(19) u_{hk}^1 is the solution of the problem (14) with $m = 1$.

Using the Lax–Milgram theorem (see [1]), one can easily check that (15), (16) are uniquely solvable.

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Let u_{hk}, w_{hk}, z_{hk} be functions with values in V_h, defined by

$$u_{hk}(t) = u_{hk}^{2i}, \tag{20}$$

$$w_{hk}(t) = u_{hk}^{2i-1}, \tag{21}$$

$$y_{hk}(t) = u_{hk}^{2i-1,\theta} \tag{22}$$

in all cases for $t \in (2(i-1)k, 2ik]$ and $i = 1, \ldots, N/2$,

$$\begin{cases} z_{hk}(2ik) = u_{hk}^{2i} & \text{for} \quad i = 0, 1, \ldots, N/2, \\ z_{hk}|_{[2(i-1)k,\, 2ik]} & \text{is a linear function for } i = 1, \ldots, N/2, \\ z_{hk}(t) = 0 & \text{for} \quad t \notin [0, T], \end{cases} \tag{23}$$

where $\{u_{hk}^m\}_{m=0}^N$ is a solution of the problem (15) or (16) with u_{hk}^0, u_{hk}^1 given by (18) and (19).

Now we want to derive some *à priori* estimates of solutions of (15), (16) and to prove some lemmas, which provide a few interesting properties of functions defined by (20)–(23) and allow us to prove the main theorem of this paper. Let us begin from the following lemma.

LEMMA 1. *If $\{u_{hk}^m\}_{m=0}^N$ is a solution of the problem* (15), *then there exist positive constants C_1, C_2 and C_3 (depending only on f, u_0 and ν), such that for all h and k the following inequalities are true*:

$$\max_{0 \leqslant m \leqslant N} |u_{hk}^m|^2 \leqslant C_1, \tag{24}$$

$$\sum_{m=1}^{N} k \|u_{hk}^m\|^2 \leqslant C_2, \tag{25}$$

$$\sum_{m=1}^{N-1} |u_{hk}^{m+1} - u_{hk}^{m-1}|^2 \leqslant C_3. \tag{26}$$

Proof. If we take $v_h = u_{hk}^{m+1}$ in (15), we obtain the equality

$$|u_{hk}^{m+1}|^2 - |u_{hk}^{m-1}|^2 + |u_{hk}^{m+1} - u_{hk}^{m-1}|^2 + 4k\nu \|u_{hk}^{m+1}\|^2 = 4k(f^m, u_{hk}^{m+1});$$

but using the Cauchy–Schwarz inequality, (11) and ε-inequality we have

$$4k(f^m, u_{hk}^{m+1}) \leqslant k\left(3\nu \|u_{hk}^{m+1}\|^2 + \frac{4C^2}{3\nu} |f^m|^2\right),$$

and thus

$$|u_{hk}^{m+1}|^2 - |u_{hk}^{m-1}|^2 + |u_{hk}^{m+1} - u_{hk}^{m-1}|^2 + k\nu \|u_{hk}^{m+1}\|^2 \leqslant \frac{4C^2 k}{3\nu} |f^m|^2.$$

Summing these expressions for $m = 1, \ldots, r-1$ we get

$$|u_{hk}^r|^2 + |u_{hk}^{r-1}|^2 + \sum_{m=1}^{r-1} |u_{hk}^{m+1} - u_{hk}^{m-1}|^2 + \nu \sum_{m=1}^{r-1} k \|u_{hk}^{m+1}\|^2 \leqslant \frac{4C^2}{3\nu} \int_0^T |f(s)|^2 ds + |u_{hk}^0|^2 + |u_{hk}^1|^2$$

for every $1 \leqslant r \leqslant N$.

From this inequality and from the known bounds for u_{hk}^0, u_{hk}^1 (see [6]), we obtain (24), (25) and (26).

LEMMA 2. *Let us assume that $\theta > \frac{1}{4}$. If $\{u_{hk}^m\}_{m=0}^N$ is a solution of the problem* (16), *then there exist positive constants C_4 and C_5 such that for all h and k the following inequalities are true*:

$$\max_{0 \leqslant m \leqslant N} |u_{hk}^m|^2 \leqslant C_4, \tag{27}$$

$$\sum_{m=1}^{N-1} k \|u_{hk}^{m\theta}\|^2 \leqslant C_5. \tag{28}$$

If in addition

$$kS^2(h) \leqslant \text{const}, \quad kR^2(h) \leqslant \text{const}, \tag{29}$$

then there exists a positive constant C_6 such that for all h and k the following inequality is true:

$$\sum_{m=1}^{N-1} |u_{hk}^{m+1} - u_{hk}^{m-1}|^2 \leqslant C_6. \tag{30}$$

Proof. If we take the $v_h = u_{hk}^{m\theta}$ in (16), we obtain the equality

$$(1-2\theta)\{|u_{hk}^{m+1/2}|^2 - |u_{hk}^{m-1/2}|^2\} + (\theta - \tfrac{1}{4})\{|u_{hk}^{m+1}|^2 - |u_{hk}^{m-1}|^2\} + k\nu \|u_{hk}^{m\theta}\|^2 = k(f^m, u_{hk}^{m\theta}),$$

where

$$u_{hk}^{m+1/2} = \tfrac{1}{2}(u_{hk}^{m+1} + u_{hk}^m), \quad u_{hk}^{m-1/2} = \tfrac{1}{2}(u_{hk}^m + u_{hk}^{m-1}),$$

but using the Cauchy–Schwarz inequality, (11), ε-inequality and summing for $m = 1, \ldots, r-1$ we have

$$(\theta - \tfrac{1}{4})\{|u_{hk}^r|^2 + |u_{hk}^{r-1}|^2\} + (1-2\theta)|u_{hk}^{r-1/2}|^2 + \frac{\nu}{2} \sum_{m=1}^{r-1} k \|u_{hk}^{m\theta}\|^2 \leqslant \frac{C^2 k}{2\nu} \sum_{m=1}^{r-1} |f^m|^2 + (\theta - \tfrac{1}{4})\{|u_{hk}^0|^2 + |u_{hk}^1|^2\} + (1-2\theta)|u_{hk}^{1/2}|^2.$$

Now we see that the following inequality is valid

$$\min(\tfrac{1}{4}, \theta-\tfrac{1}{4})\{|u_{hk}^r|^2+|u_{hk}^{r-1}|^2\}+\frac{\nu}{2}\sum_{m=1}^{r-1} k\|u_{hk}^{m\theta}\|^2$$

$$\leqslant \frac{C^2}{2\nu}\int_0^T |f(s)|^2 ds+(\theta+\tfrac{3}{4})\{|u_{hk}^0|^2+|u_{hk}^1|^2\}.$$

From this inequality and from the known bounds for u_{hk}^0 and u_{hk}^1 we obtain (27) and (28).

In order to prove the boundedness of $|u_{hk}^{m+1}-u_{hk}^{m-1}|^2$, we take $v_h = 4k(u_{hk}^{m+1}-u_{hk}^{m-1})$ in (16) and we write

$$2|u_{hk}^{m+1}-u_{hk}^{m-1}|^2$$
$$= -4k((u_{hk}^m, u_{hk}^{m+1}-u_{hk}^{m-1}))+4k(f^m, u_{hk}^{m+1}-u_{hk}^{m-1})-4kb(u_{hk}^m, u_{hk}^m, u_{hk}^{m+1}-u_{hk}^{m-1}).$$

An application of the Cauchy–Schwarz inequality, ε-inequality and (12), (13) allows us to estimate the right-hand side of this equation and to write the following inequality:

$$|u_{hk}^{m+1}-u_{hk}^{m-1}|^2 \leqslant 8\nu^2k^2S^2(h)\|u_{hk}^{m\theta}\|^2+8C^2k^2R^2(h)|u_{hk}^m|^2\|u_{hk}^{m\theta}\|^2+8k^2|f^m|^2.$$

Summing these relations for $m = 1, \ldots, N-1$ and using (28), (29) we get

$$\sum_{m=1}^{N-1}|u_{hk}^{m+1}-u_{hk}^{m-1}|^2 \leqslant 8\nu^2C_5kS^2(h)+8C^2C_4C_5kR^2(h)+8T\int_0^T|f(s)|^2ds.$$

Hence the required result follows.

Following [4] and [6], let us define for $0<\gamma<1/4$ the Hilbert space $\mathscr{H}^\gamma(R; V, H)$ as the space of functions $v\in L^2(R; V)$ such that $D_t^\gamma v$ belongs to $L^2(R; H)$, where $D_t^\gamma v$ denotes the derivative in t of order γ of v, defined by the formula

$$\widehat{D_t^\gamma v(\tau)} = (2\pi i\tau)^\gamma \hat{v}(\tau)$$

($\hat{f}$ denotes the Fourier transform of f: $R\to V$).

The norm in $\mathscr{H}^\gamma(R; V, H)$ is defined by

$$\|v\|_{\mathscr{H}^\gamma(R;V,H)} = \{\|v\|^2_{L^2(R;V)}+\||\tau|^\gamma\hat{v}\|^2_{L^2(R;H)}\}^{1/2}.$$

Now we are able to state the next lemma.

Lemma 3. *If the functions z_{hk}: $R\to V_h$ are defined by* (23) *with* $\{u_{hk}^m\}_{m=0}^N$ *given by* (15) *(or by* (16)*), then the family z_{hk} forms a bounded set in the Hilbert space* $\mathscr{H}^\gamma(R; V, H)$.

Proof. See [6], pp. 275–277 and [4], pp. 77–79, where the method which can be used to prove this lemma is described in all details.

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Now we can summarize the results of this paper in the following theorem.

Theorem 1. *Let us assume that $2\leqslant n\leqslant 4$ and that we have a sequence of parameters $\{(h, k)\}$ such that $h\to 0$, $k\to 0$. If u is a solution of the problem* (2)–(6) *and if $\{u_{hk}^m\}_{m=0}^N$ is a family of solutions of the problem* (15), (18), (19) *for all natural even numbers N and all $h\in G$, then there exists a subsequence $\{(h', k')\}$ of $\{(h, k)\}$ such that for functions u_{hk} defined by* (20) *the following convergence results are true:*

(31) $u_{h'k'}\to u$ *weakly* in* $L^\infty(0, T; H)$,

(32) $u_{h'k'}\to u$ *strongly in* $L^2(0, T; H)$,

(33) $u_{h'k'}\to u$ *weakly in* $L^2(0, T; V)$,

for $h'\to 0$, $k'\to 0$.

Proof. The definitions (20), (21), (23) of functions u_{hk}, w_{hk}, z_{hk} yield the equation

(34) $\forall v_h\in V_h$

$$\frac{d}{dt}(z_{hk}(t), v_h)+\nu((u_{hk}(t), v_h))+b(w_{hk}(t), u_{hk}(t), v_h) = (f_k(t), v_h)$$

for $t\in(0, T)$, where $f_k(t) = f^{2j-1}$ for $t\in(2(j-1)k, 2jk]$ and $j = 1, \ldots, N/2$. It follows immediately from Lemma 1 that the family of functions u_{hk} forms a bounded set both in $L^2(0, T; V)$ and in $L^\infty(0, T; H)$, and thus there exists a subsequence $\{(h', k')\}$ of $\{(h, k)\}$ such that

(35) $u_{h'k'}\to u$ weakly in $L^2(0, T; V)$,

(36) $u_{h'k'}\to u$ weakly in $L^\infty(0, T; H)$,

(37) $u_{h'k'}\to u$ weakly in $L^2(0, T; H)$,

with some $u\in L^2(0, T; V)\cap L^\infty(0, T; H)$.

Lemma 3 implies that the family of functions z_{hk} forms a bounded set in the space $\mathscr{H}^\gamma(R; V, H)$ and thus, by the compactness theorem (see [4], pp. 61–62 and [6], pp. 215–220), there exists a subsequence $\{(h', k')\}$ of $\{(h, k)\}$ such that

$$z_{h'k'}\to z \text{ strongly in } L^2(0, T; H),$$

for some $z\in L^2(0, T; H)$.

Since $u_{hk}-z_{hk}\to 0$ and $u_{hk}-w_{hk}\to 0$ strongly in $L^2(0, T; H)$, it is easy to see that

(38) $u_{h'k'}\to u$ strongly in $L^2(0, T; H)$,

(39) $z_{h'k'}\to u$ strongly in $L^2(0, T; H)$,

(40) $w_{h'k'}\to u$ strongly in $L^2(0, T; H)$.

It can be also checked (in the same fashion as in [6], p. 284), that

(41) $f_k\to f$ strongly in $L^2(0, T; H)$, with $k\to 0$.

Now we are able to pass to the limit in (34) with $h', k'\to 0$ in the sense of convergence in $D'(0, T)$. Using (35), (38)–(41) and (9) we see that for all $h_0\in G$ and all test functions $\varphi \in D(0, T)$, the following results are true:

$$\int_0^T \frac{d}{dt}(w_{h'k'}(t), v_{h_0})\varphi(t)\,dt \to -\int_0^T (u(t), v_{h_0})\varphi'(t)\,dt,$$

$$\int_0^T ((u_{h'k'}(t), v_{h_0}))\varphi(t)\,dt \to \int_0^T ((u_{h'k'}(t), v_{h_0}))\varphi(t)\,dt,$$

$$\int_0^T (f_{k'}(t), v_{h_0})\varphi(t)\,dt \to \int_0^T (f(t), v_{h_0})\varphi(t)\,dt.$$

It is known (see [4], p. 73), that $w_{h'k',i}u_{h'k',j} \to w_i u_j$ weakly in $L^2(0, T; L^{p/2}(\Omega))$ for $p = 1/2-1/2n$.

An application of this fact and of assumption (10) yields the convergence

$$\int_0^T b(w_{h'k'}(t), u_{h'k'}(t), v_{h_0})\varphi(t)\,dt \to \int_0^T b(u(t), u(t), v_{h_0})\varphi(t)\,dt.$$

This implies that u occurring in (35), (36), (38) satisfies condition (5). It can be checked by the method used in [4] and [6] that u satisfies also condition (6). Thus we can conclude that conditions (31)–(33) are satisfied for the weak solution u of the Navier–Stokes equation.

THEOREM 2. *Let us assume that $2 \leqslant n \leqslant 4$ and that we have $\theta > \frac{1}{4}$ and a sequence of parameters $\{(h, k)\}$ satisfying the conditions* (29) *and such that $h \to 0$, $k \to 0$.*

If u is a solution of the problem (2)–(6) *and $\{u_{hk}^m\}_{m=0}^N$ is a family of solutions of the problem* (16), (18), (19) *for all natural even numbers N and all $h \in G$, then there exists a subsequence $\{(h', k')\}$ of $\{(h, k)\}$ such that for functions u_{hk} defined by* (20) *the convergence results* (31)–(33) *are true.*

The theorem is proved in the same way as Theorem 1 (we must only replace (20) by (22) and use Lemma 2 instead of Lemma 1).

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РЕШЕНИЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО ТИПА МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

АЛОЙЗ НЕМЕТЫ

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1. Введение

В настоящей работе метод конечных элементов применяется в решении смешанной задачи для параболических дифференциальных уравнений с частными производными 2-го порядка — в уравнении теплопроводности. Как и при решении проблем линейной вязкоупругости [4, 5, 6, 7], так и в данном случае можно применить преобразование Лапласа и потом решать присоединенную краевую задачу методом конечных элементов.

Обратное преобразование Лапласа является очень сложным и его можно определить только численным путем. Характер обратного преобразования виден из разложения присоединенного решения на частные дроби, но практически для численного решения применение этого метода невозможно. В данном случае подходящим численным методом для определения обратного преобразования Лапласа является метод разложения искомого решения в ряд Дирихле.

2. Формулировка проблемы

Пусть Ω — ограниченная область, $\Omega \subset E_2$, где E_2 представляет двумерное эвклидово пространство. Пусть S граница области Ω. Уравнение теплопроводности примет вид

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{2}\sum_{k=1}^{2} \frac{\partial}{\partial x_i} a_{ik}(X) \frac{\partial u}{\partial x_k} = f(X), \tag{2.1}$$

где $X = (x_1, x_2)$, $a_{ik}(X)$, $i, k = 1, 2$ удовлетворяют для $X \in \Omega$ неравенству