и вычислим детерминант этой матрицы

$$|M| = \left|\frac{\lambda_j}{\lambda_i+\lambda_j}\right|_{i,j=1,2,\dots,n} = \left\{\prod_{k=1}^{n}\lambda_k\right\}\left|\frac{1}{\lambda_i+\lambda_j}\right|_{i,j=1,2,\dots,n}.$$

Применяя предыдущую лемму, получим

$$|M| = \left\{\prod_{k=1}^{n}\lambda_k\right\}\left\{\prod_{i=1}^{n}\prod_{k=1}^{n}\frac{1}{\lambda_i+\lambda_k}\right\}\left\{\prod_{i=1}^{n}\prod_{k=i+1}^{n}(\lambda_i-\lambda_k)^2\right\} \neq 0$$

что и требовалось доказать.

Коэффициенты λ_i ряда Дирихле (4.2) нужно заранее определить из интервала $(0, \infty)$. Можно также определить коэффициенты λ_i путем минимизации квадрата ошибки E^2 с учетом λ_i для $i = 1, 2, \dots, n$, но эта проблема приводит к нелинейной системе алгебраических уравнений, которую практически невозможно решить.

5. Заключение

Метод конечных элементов большей частью применяется для решения краевых задач для дифференциальных уравнений эллиптического типа. В примененном в данной работе методе решения смешанных задач для уравнений параболического типа можно использовать разработанные уже программы и известные алгоритмы для решения краевых задач дла эллиптических уравнений 2-го порядка.

При применении треугольных и прямоугольных элементов доказана сходимость решения присоединенной проблемы [3, 9, 10].

Для обратного преобразования Лапласа показан подходящий метод разложения решения в ряд Дирихле, который приносит очень хорошие результаты в области задач линейной вязкоупругости [4, 5, 6, 7]. Для более точного определения обратного преобразования, главным образом для малых значений времени t, нужно аппроксимировать решение рядом (4.3) и найти алгоритм для определения коэффициентов этого ряда.

При применении пространственных элементов можно аналогично решить также трехмерное уравнение теплопроводности.

В данной работе правая часть уравнения (2.1) не зависит от времени t. При достаточной гладкости функции $f(X, t)$ можно тем же способом решить также эту наидалее общую проблему, но при обратном преобразовании Лапласа невозможно выразить решение в виде (4.1), потому что вектор правых частей F не будет линейной функцией параметра p.

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PROJECTION METHODS IN THE APPROXIMATE SOLUTION OF THE EIGENVALUE PROBLEM IN A HILBERT SPACE

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1. Introduction

Let H be real or complex Hilbert space with the scalar product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$.

Let Θ be a subset of the real interval $(0, 1]$ with an accumulation point at zero.

In what follows V_h will denote a linear subspace of H of finite dimension d_h, where $d_h \to \infty$ when $h \to 0$.

We are interested in the approximation of the spectrum of some class of linear operators defined, in general, on some subspace of H, by means of consecutive projections onto appropriately chosen subspaces $V_h \subset H$, when $h \to 0$.

Our investigations are based on the classical so-called Perturbation Theorem [1].

Let us formulate this theorem in the case of a compact operator T,

$$T\colon H \to H.$$

It is very well known that the spectrum of a compact operator is a finite or infinite sequence of eigenvalues with possibly only one accumulation point zero. Denote by $\sigma(T) = \{\lambda_\nu\}_{\nu=1,2,\dots}$ the spectrum of T. Let ε and δ be sufficiently small positive numbers. Denote by K_δ the disc of radius δ centred at zero, and by K_ε^ν the discs of radius ε, centred at λ_ν. Given δ, there is only a finite number of λ's outside K_δ.

Assume that:

—the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{N_\delta}$ lie outside K_δ;

—$K_\delta \cap K_\varepsilon^l = \emptyset$ and $K_\varepsilon^k \cap K_\varepsilon^l = \emptyset$ for $k \neq l$, $k, l = 1, 2, \dots, N_\delta$.

Then we have

THE PERTURBATION THEOREM. *Let a family of linear continuous operators $\{T_h\}_{h\in\Theta}$ be given*

$$T_h\colon H \to H,$$

such that

$$\|T_h - T\| \to 0 \quad \text{as} \quad h \to 0.$$

For any ε and δ there is $h_{\varepsilon\delta} > 0$ such that for every $0 < h < h_{\varepsilon\delta}$, $h \in \Theta$

—each disc K_ε^j, $j = 1, 2, \ldots, N_\delta$, contains exactly r_j eigenvalues of T_h (counting their multiplicities), where r_j is the algebraic multiplicity of λ_j;

—all eigenvalues of T_h which are contained in none of the discs K_ε^j, $j = 1, 2, \ldots, N_\delta$, lie in the disc K_δ. ■

Here we denote by $\|\cdot\|_{XY}$ the norm of a linear map defined on the space X with values in the space Y.

We write $\|\cdot\|_X$ instead of $\|\cdot\|_{XX}$, or simply $\|\cdot\|$, if the space is fixed and no confusion can arise.

2. Projections from H onto finite dimensional subspaces

A bounded linear map π: $H \to V$, where V is an arbitrary linear subspace of H, is a *projection* iff

(2.1) $$\pi\pi = \pi.$$

A projection π is called *orthogonal* iff

$$\forall x \in H \, \forall \xi \in V \quad (x - \pi x, \xi) = 0.\text{[1]}$$

It is easy to verify that:

—a projection π is orthogonal iff $\pi = \pi^*$;

—for any projection π we have $\|\pi\| \geqslant 1$, and if π is orthogonal then $\|\pi\| = 1$.

LEMMA 2.1. *Each linear map π_h: $H \to V_h$ onto a d_h-dimensional subspace $V_h \subset H$ is of the form*

(2.2) $$\pi_h x = \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h),$$

where φ_j^h, $j = 1, 2, \ldots, d_h$, are linearly independent elements of some basis of V_h and ψ_j^h, $j = 1, 2, \ldots, d_h$, are some linearly independent elements of H. Given a basis φ_j^h, $j = 1, 2, \ldots, d_h$, the elements ψ_j^h, $j = 1, 2, \ldots, d_h$, are uniquely determined by π_h.

Proof. Observe that

$$\forall x \in H \quad \pi_h x = \sum_{j=1}^{d_h} c_j^h(x) \varphi_j^h$$

and hence

$$\begin{bmatrix} c_1^h(x) \\ \vdots \\ c_{d^h}^h(x) \end{bmatrix} = G_{\varphi^h}^{-1} \begin{bmatrix} (\pi x, \varphi_1^h) \\ \vdots \\ (\pi x, \varphi_{d_h}^h) \end{bmatrix},$$

where $G_{\varphi^h} = ((\varphi_j^h, \varphi_i^h))_{i,j=1,2,\ldots,d_h}$ is the Gramm matrix of the system $\varphi_1^h, \ldots, \varphi_{d_h}^h$.

[1] Where we denote by $(\cdot, \cdot)$ the scalar product in H.

This means that $c_j^h(x)$, $j = 1, 2, \ldots, d_h$, are linear continuous functionals on H and hence

$$c_j^h(x) = (x, \psi_j^h), \quad j = 1, 2, \ldots, d_h,$$

where $\psi_j^h \in H$ are uniquely determined elements of H.

Now suppose that for some complex numbers $\mu_1, \ldots, \mu_{d_h}$

$$\sum_{j=1}^{d_h} |\mu_j| > 0$$

and

$$\sum_{j=1}^{d_h} \mu_j \psi_j^h = 0.$$

Put $\gamma = [\mu_1, \ldots, \mu_{d_h}]^{\mathrm{T}}$ and $c^h(x) = [c_1^h(x), \ldots, c_{d_h}^h(x)]^{\mathrm{T}}$; then

$$\forall x \in H \quad 0 = \Big(x, \sum_{j=1}^{d_h} \mu_j \psi_j^h\Big)_H = \gamma^* c^h(x) = \gamma^* G_{\varphi^h}^{-1} \begin{bmatrix} (\pi_h x, \varphi_1^h) \\ \vdots \\ (\pi_h x, \varphi_{d_h}^h) \end{bmatrix}.$$

Denoting $[\bar{\lambda}_1, \ldots, \bar{\lambda}_{d_h}] = \gamma^* G_{\varphi^h}^{-1}$ we get

$$\forall x \in H \quad \Big(\pi_h x, \sum_{j=1}^{d_h} \lambda_j \varphi_j^h\Big)_H = 0 \quad \text{or} \quad \forall \xi \in V_h \quad \Big(\xi, \sum_{j=1}^{d_h} \lambda_j \varphi_j^h\Big)_H = 0,$$

which is impossible because $[\bar{\lambda}_1, \ldots, \bar{\lambda}_{d_h}] \neq 0$ and $\{\varphi_j^h\}_{j=1,2,\ldots,d_h}$ is a basis of V_h. ■

LEMMA 2.2. *A linear map* (2.2) *is a projection iff the systems $\{\varphi_j^h\}_{j=1,2,\ldots,d_h}$ and $\{\psi_j^h\}_{j=1,2,\ldots,d_h}$ are mutually biorthonormal, i.e., iff*

$$(\varphi_k^h, \psi_l^h) = \delta_{kl}, \quad k, l = 1, 2, \ldots, d_h.$$

Proof. (a) Let $(\varphi_k^h, \psi_l^h) = \delta_{kl}$. Then

$$\pi_h \pi_h x = \sum_{j=1}^{d_h} \varphi_j^h (\pi_h x, \psi_j^h) = \sum_{j=1}^{d_h} \sum_{l=1}^{d_h} \varphi_j^h (\varphi_l^h \psi_j^h)(x, \psi_l^h) = \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h) = \pi_h x,$$

i.e. π_h is a projection.

(b) Let $\pi_h \pi_h = \pi_h$. Then

$$\forall x \in H \quad \sum_{j=1}^{d_h} \sum_{l=1}^{d_h} (\varphi_l^h, \varphi_j^h)(x, \psi_l^h) \varphi_j^h = \sum_{j=1}^{d_h} (x, \psi_j^h) \varphi_j^h.$$

Since the system $\psi_1^h, \ldots, \psi_{d_h}^h$ is linearly independent, we can choose elements $y_k \in H$, $k = 1, 2, \ldots, d_h$, such that

$$(y_k, \psi_l^h) = \delta_{kl}, \quad k, l = 1, 2, \ldots, d_h.$$

Now taking $x = y_k$, $k = 1, 2, \ldots, d_h$, we get

$$\sum_{l=1}^{d_h} (\varphi_l^h, \psi_j^h) \delta_{kl} = \delta_{kj},$$

i.e.

$$(\varphi_k^h, \psi_j^h) = \delta_{kj}, \quad k, j = 1, 2, \ldots, d_h. \ \blacksquare$$

LEMMA 2.3. *Let π_h be a projection $\pi_h\colon H \xrightarrow[\text{onto}]{} V_h$. Then π_h can be split into the superposition of two maps p_h and r_h,*

$$\pi_h = p_h r_h, \tag{2.3}$$

where $p_h\colon X_h \xrightarrow[\text{into}]{} H$ is a one-to-one linear map from a d_h-dimensional vector space X_h into H, and $r_h\colon H \xrightarrow[\text{onto}]{} X_h$ is a linear map from H into X_h. Moreover, $r_h p_h = I_h$ is the identity map of X_h. Given a basis $\varphi_1^h, \ldots, \varphi_{d_h}^h$, the pair p_h, r_h is uniquely determined.

Proof. Put

$$r_h x \underset{\text{df}}{=} [(x, \psi_1^h), \ldots, (x, \psi_{d_h}^h)]^{\mathrm{T}} \in X_h, \tag{2.4}$$

$$p_h x_h \underset{\text{df}}{=} \sum_{j=1}^{d_h} \varphi_j^h x_j^h, \tag{2.5}$$

where

$$x_h = [x_1^h, \ldots, x_{d_h}^h]^{\mathrm{T}} \in X_h.$$

Then $\pi_h = p_h r_h$ and

$$\forall x_h \in X_h \quad r_h p_h x_h = \Big[\Big(\sum_{j=1}^{d_h} \varphi_j^h x_j^h, \psi_1^h\Big), \ldots, \Big(\sum_{j=1}^{d_h} \varphi_j^h x_j^h, \psi_{d_h}^h\Big)\Big]^{\mathrm{T}} = [x_1^h, \ldots, x_{d_h}^h]^{\mathrm{T}} = x_h,$$

because the systems $\varphi_1^h, \ldots, \varphi_{d_h}^h$ and $\psi_1^h, \ldots, \psi_{d_h}^h$ are biorthonormal. $\blacksquare$

LEMMA 2.4. *Let*

$$\pi_h x = \sum_{j=1}^{d_h} (x, \psi_j^h)\varphi_j^h.$$

Then

$$\pi_h^* x = \sum_{j=1}^{d_h} (x, \varphi_j^h)\psi_j^h.$$

Proof.

$$\forall x, y \in H \quad (\pi_h x, y) = \Big(\sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h), y\Big) = \sum_{j=1}^{d_h} (x, \psi_j^h)(\varphi_j^h, y)$$
$$= \sum_{j=1}^{d_h} (x, \psi_j^h)\overline{(y, \varphi_j^h)} = \Big(x, \sum_{j=1}^{d_h} \psi_j^h (y, \varphi_j^h)\Big) = (x, \pi_h^* y),$$

i.e.

$$\pi_h^* x = \sum_{j=1}^{d_h} (x, \varphi_j^h)\psi_j^h. \ \blacksquare$$

LEMMA 2.5. *π_h is the orthogonal projection onto V_h, iff both mutually biorthonormal systems $\varphi_1^h, \ldots, \varphi_{d_h}^h$ and $\psi_1^h, \ldots, \psi_{d_h}^h$ are contained in V_h. In this case*

$$G_{\psi^h} = G_{\varphi^h}^{-1} \tag{2.6}$$

and

$$[\psi_1^h, \ldots, \psi_{d_h}^h] = [\varphi_1^h, \ldots, \varphi_{d_h}^h] G_{\varphi^h}^{-1}, \tag{2.7}$$

where G_{φ^h} and G_{ψ^h} are the corresponding Gramm matrices for the systems $\varphi_1^h, \ldots, \varphi_{d_h}^h$ and $\psi_1^h, \ldots, \psi_{d_h}^h$.

Proof. (a) Let π_h be the orthogonal projection onto V_h. Then $\pi_h^* = \pi_h$, i.e.

$$\forall x \in H \quad \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h) = \sum_{j=1}^{d_h} \psi_j^h (x, \varphi_j^h).$$

In particular, for $x = \varphi_k^h$ we have

$$\sum_{j=1}^{d_h} \varphi_j^h (\varphi_k^h, \psi_j^h) = \varphi_k^h = \sum_{j=1}^{d_h} \psi_j^h (\varphi_k^h, \varphi_j^h) \tag{2.8}$$

and for $x = \psi_k^h$

$$\sum_{j=1}^{d_h} \varphi_j^h (\psi_k^h, \psi_j^h) = \psi_k^h = \sum_{j=1}^{d_h} \psi_j^h (\psi_k^h, \varphi_j^h). \tag{2.9}$$

Using the above relations we obtain

$$\sum_{j=1}^{d_h} \sum_{l=1}^{d_h} \psi_l^h (\varphi_j^h, \varphi_l^h)(\psi_k^h, \psi_j^h) = \psi_k^h \quad \text{and} \quad \sum_{j=1}^{d_h} (\varphi_j^h, \varphi_l^h)(\psi_k^h, \psi_j^h) = \delta_{lk},$$

i.e.

$$G_{\varphi^h} G_{\psi^h} = I. \tag{2.10}$$

The relation (2.6) follows from (2.10), and (2.7) from (2.9). Hence

$$\{\psi_1^h, \ldots, \psi_{d_h}^h\} \subset V_h.$$

(b) Conversely, if $\{\psi_1^h, \ldots, \psi_{d_h}^h\} \subset V_h$, we get immediately by the biorthonormality

$$\psi_k^h = \sum_{j=1}^{d_h} \alpha_{kj} \varphi_j^h,$$

where

$$\alpha = (\alpha_{kj})_{k, j = 1, 2, \ldots, d_h} = G_{\varphi^h}^{-1},$$

and hence $\alpha_{kj} = \bar{\alpha}_{jk}$. Then

$$\forall x \in H \quad \pi_h^* x = \sum_{k=1}^{d_h} \sum_{j=1}^{d_h} \alpha_{kj} \varphi_j^h (x, \varphi_k^h) = \sum_{j=1}^{d_h} \varphi_j^h \Big(x, \sum_{k=1}^{d_h} \bar{\alpha}_{kj} \varphi_k^h\Big)$$
$$= \sum_{j=1}^{d_h} \varphi_j^h \Big(x, \sum_{k=1}^{d_h} \alpha_{jk} \varphi_k^h\Big) = \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_k^h) = \pi_h x,$$

i.e. π_h is the orthogonal projection. $\blacksquare$

Let

$$\pi_h x = \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h)$$

be a family of projections onto the spaces V_h, $h \in \Theta$.

The adjoint projections π_h^* admit splitting of the form (2.3). Denote

$$\pi_h^* = q_h s_h, \tag{2.11}$$

where

$$q_h x_h = \sum_{j=1}^{d_h} \psi_j^h x_j^h; \quad x_h = [x_1^h, \ldots, x_{d_h}^h]^{\mathrm{T}} \in X_h, \tag{2.12}$$

$$s_h x = [(x, \varphi_1^h), \ldots, (x, \varphi_{d_h}^h)]^{\mathrm{T}} \in X_h, \quad x \in H. \tag{2.13}$$

We need here to introduce a norm topology in the spaces X_h.

In general, the choice of a norm in X_h depends on the problem under consideration, however the main results of later sections are independent of this particular choice.

Let us discuss briefly one of such possibilities.

In the finite dimensional space X_h we introduce the usual Cartesian scalar product

$$(x_h, y_h)_{X_h} = \sum_{j=1}^{d_h} x_j^h \bar{y}_j^h,$$

where

$$x_h = [x_1^h, \ldots, x_{d_h}^h]^{\mathrm{T}} \quad \text{and} \quad y_h = [y_1^h, \ldots, y_{d_h}^h]^{\mathrm{T}}.$$

Observe that

$$\begin{aligned} (p_h x_h, q_h y_h) &= \Big(\sum_{j=1}^{d_h} \varphi_j^h x_j^h, \sum_{k=1}^{d_h} \psi_k^h y_k^h\Big) = \sum_{j=1}^{d_h}\sum_{k=1}^{d_h} (\varphi_j^h, \psi_k^h) x_j^h \bar{y}_k^h \\ &= \sum_{k=1}^{d_h}\sum_{j=1}^{d_h} (\psi_k^h, \varphi_j^h) x_k^h \bar{y}_j^h = (q_h x_h, p_h y_h) = (x_h, y_h)_{X_h}. \end{aligned} \tag{2.14}$$

LEMMA 2.6. *Let $0 \leqslant \lambda_1^h \leqslant \ldots \leqslant \lambda_{d_h}^h$ and $0 \leqslant \mu_1^h \leqslant \ldots \leqslant \mu_{d_h}^h$ be the eigenvalues of the Gramm matrices G_{φ^h} and G_{ψ^h}, respectively.*

Then the following inequalities hold:

$$\sqrt{\lambda_1^h} \leqslant \|p_h\|_{X_h H} \leqslant \sqrt{\lambda_{d_h}^h}, \tag{2.15}$$

$$\sqrt{\mu_1^h} \leqslant \|q_h\|_{X_h H} \leqslant \sqrt{\mu_{d_h}^h}, \tag{2.16}$$

$$\|r_h\|_{H X_h} \leqslant \|q_h\|_{X_h H}, \tag{2.17}$$

$$\|s_h\|_{H X_h} \leqslant \|p_h\|_{X_h H}, \tag{2.18}$$

$$\|\pi_h\| = \|\pi_h^*\| = \|p_h r_h\| \leqslant \|p_h\|_{X_h H} \|r_h\|_{H X_h} \leqslant \sqrt{\lambda_{d_h}^h \mu_{d_h}^h}. \tag{2.19}$$

Proof. $\forall x_h \in X_h$, $x_h = [x_1^h, \ldots, x_{d_h}^h]^{\mathrm{T}}$,

$$\|p_h x_h\|_H^2 = \Big(\sum_{j=1}^{d_h} \varphi_j^h x_j^h, \sum_{k=1}^{d_h} \varphi_k^h x_k^h\Big) = \sum_{j=1}^{d_h}\sum_{k=1}^{d_h} (\varphi_j^h, \varphi_k^h) x_j^h \bar{x}_k^h = x_h^* G_{\varphi^h} x_h.$$

Since G_{φ^h} is Hermitian, we have

$$G_{\varphi^h} = U_h^* \Lambda_h U_h,$$

where

$$\Lambda_h = \begin{bmatrix} \lambda_1^h & & 0 \\ & \ddots & \\ 0 & & \lambda_{d_h}^h \end{bmatrix}$$

and U_h is some unitary matrix. Put $\xi_h = U_h x_h$; then $\xi_h^* \xi_h = x_h^* x_h$ and (2.15) follows immediately. In a similar way we obtain (2.16).

Now, $\forall x \in H$

$$\begin{aligned} \|r_h x\|_{X_h}^2 &= (p_h r_h x, q_h r_h x)_H = (\pi_h x, q_h r_h x)_H = (x, \pi_h^* q_h r_h x)_H \\ &= (x, q_h s_h q_h r_h x)_H = (x, q_h r_h x)_H \leqslant \|r_h x\|_{X_h} \|q_h\|_{X_h H} \|x\|_H \end{aligned}$$

and

$$\|r_h x\|_{X_h} \leqslant \|q_h\|_{X_h H} \|x\|_H;$$

hence (2.17) follows. Using (2.14), we get (2.18) analogously. ∎

COROLLARY 1. *If π_h, $h \in \Theta$, are orthogonal, then by Lemma 2.5 we have $G_{\psi^h} = G_{\varphi^h}^{-1}$ and hence*

$$\mu_1^h = \frac{1}{\lambda_{d_h}^h}, \quad \ldots, \quad \mu_{d_h}^h = \frac{1}{\lambda_1^h}.$$

In this case

$$\sqrt{\lambda_1^h} \leqslant \|p_h\|_{X_h H} \leqslant \sqrt{\lambda_{d_h}^h}, \tag{2.20}$$

$$\|r_h\|_{H X_h} \leqslant \frac{1}{\sqrt{\lambda_1^h}}, \tag{2.21}$$

$$\|\pi_h\| \leqslant \sqrt{\frac{\lambda_{d_h}^h}{\lambda_1^h}}. \tag{2.22}$$

COROLLARY 2. (a) *If the eigenvalues of G_{φ^h} and G_{ψ^h} are uniformly bounded with respect to $h \in \Theta$, then the families*

$$\{p_h\}_{h\in\Theta}, \quad \{r_h\}_{h\in\Theta}, \quad \{q_h\}_{h\in\Theta}, \quad \{s_h\}_{h\in\Theta}, \quad \{\pi_h\}_{h\in\Theta} \tag{2.23}$$

are equicontinuous.

(b) *If $\{\pi_h\}_{h\in\Theta}$ are orthogonal projections and the eigenvalues of G_{φ^h} are uniformly bounded and bounded out from zero with respect to $h \in \Theta$, then the families (2.23) are equicontinuous.*

3. Discretisation of the space H [2], approximation, stability

Consider the following family of triads

$$\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$$

where X_h are linear real or complex normed vector spaces of dimension d_h ($d_h \to \infty$ when $h \to 0$); p_h are linear continuous one-to-one maps $p_h\colon X_h \to H$ (called prolongations); r_h are linear continuous maps $r_h\colon H \xrightarrow[\text{onto}]{} X_h$ (called restrictions).

We assume that $\forall h \in \Theta$, $r_h p_h = I_h$ is the identity on X_h. Observe that if $\pi_h = p_h r_h$ then $\forall h \in \Theta$, π_h is the projection from H onto $V_h = \pi_h H$. We assume that the family of projections $\{\pi_h\}_{h\in\Theta}$ is uniformly bounded.

If the above conditions are satisfied, then $\mathfrak{A}$ is called a *discretisation of the space* H (see also Aubin [2]).

Let $\mathfrak{A}$ be a discretisation of the space H. We say that $\mathfrak{A}$ *approximates the space* H iff there exists a dense subset $Z \subset H$ such that

$$\forall z \in Z \quad \|(\pi_h - I)z\| \to 0 \tag{3.1}$$

when $h \to 0$ (see [3]).

We say that a discretisation $\mathfrak{A}$ of the space H is *stable* iff the families

$$\{p_h\}_{\substack{h\leqslant h_0\\ h\in\Theta}}, \qquad \{r_h\}_{\substack{h\leqslant h_0\\ h\in\Theta}}$$

are equicontinuous for some fixed constant $h_0 \leqslant 1$.

Corollaries 1 and 2 to Lemma 2.6 give the following sufficient condition of stability of a discretisation $\mathfrak{A}$ for the norms in X_h, introduced in § 2.

Assume that $\exists h_0 > 0\ \exists K > 0\ \forall h \in \Theta,\ h < h_0$,

$$\begin{aligned} &\|G_{\varphi^h}\| < K,\\ &\|G_{\psi^h}\| < K, \end{aligned} \tag{3.2}$$

where $\|\cdot\|$ is an arbitrary matrix norm. Then the discretisation $\mathfrak{A}$ of the space H is stable. Here we put

$$\varphi_j^h = p_h e_j^h,$$

where e_j^h, $j = 1, 2, \ldots, d_h$, are the unit coordinate vectors of the space X_h. Then ψ_j^h, $j = 1, 2, \ldots, d_h$, are uniquely determined (Lemma 2.1).

LEMMA 3.1. *If a discretisation* $\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$ *is stable, then the family*

$$\{p_h^{-1}\}_{\substack{h\in\Theta\\ h<h_0}}$$

is equicontinuous, for some fixed h_0.

Proof. Put

$$V_h = p_h(X_h) \subset H;$$

then

$$r_h|_{V_h} = p_h^{-1}$$

and hence

$$\exists h_0\, \exists K > 0\, \forall h \in \Theta \quad h \leqslant h_0 \leftrightarrow \|p_h^{-1}\|_{HX_h} \leqslant \|r_h\|_{HX_h} \leqslant K. \ \blacksquare$$

We say that a *family* $\{V_h\}_{h\in\Theta}$ *of subspaces* V_h (2) *of the space* H *approximates the space* H iff there exists a dense subset $Z \subset H$ satisfying the following condition:

$\forall z \in Z$ there exist a family of elements $\{v_h(z)\}_{h\in\Theta}$ such that

$$v_h(z) \in V_h \quad \text{and} \quad \|z - v_h(z)\| \to 0 \quad \text{when} \quad h \to 0$$

(see [3]).

LEMMA 3.2. *Let a family* $\{V_h\}_{h\in\Theta}$ *of subspaces of* H *approximate the space* H. *Let* $\{\pi_h\}_{h\in\Theta}$ *be a family of uniformly bounded projections*

$$\pi_h\colon H \xrightarrow[\text{onto}]{} V_h,$$

$$\|\pi_h\| < K \quad (K \text{ does not depend on } h).$$

Then the discretisation $\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$, *where* $\pi_h = p_h r_h$ (see Lemma 2.3), *approximates the space* H.

Proof. It is enough to show that

$$\forall x \in H \quad \|\pi_h x - x\| \to 0 \quad \text{when} \quad h \to 0.$$

Let $x \in H$ and $\|x - z_s\| \to 0$, $s \to 0$, $z_s \in Z$. Then

$$x - \pi_h x = x - z_s + z_s - v_h(z_s) + \pi_h\big(v_h(z_s) - z_s\big) + \pi_h(z_s - x)$$

since $v_h(z_s) = \pi_h v_h(z_s)$, π_h is a projection onto V_h, and

$$\|x - \pi_h x\| \leqslant \|x - z_s\| + \|z_s - v_h(z_s)\| + K[\|v_h(z_s) - z_s\| + \|z_s - x\|].$$

For any $\varepsilon > 0$ we can first choose s such that $\|x - z_s\| < \varepsilon$ and then, for this fixed s, we choose h_0 such that for any $h \in \Theta$, $h < h_0$ the relation $\|z_s - v_h(z_s)\| < \varepsilon$ is satisfied. Finally we get

$$\|x - \pi_h x\| \leqslant \varepsilon + \varepsilon + 2K\varepsilon \quad \text{if} \quad 0 < h < h_0,\ h_0 \in \Theta. \ \blacksquare$$

LEMMA 3.3. *Let* Z *be a dense subset of* H. *If*

$$\forall z \in Z \quad (\pi_h - I)z \to 0 \quad \textit{when} \quad h \to 0,$$

then the family of subspaces $\{Y_h\}_{h\in\Theta}$, $Y_h = \pi_h(H)$, *approximates the space* H.

Proof. It is enough to take $y_h = \pi_h z$, $y_h \in Y_h$, $z \in Z$. Then

$$\|y_h - z\| = \|\pi_h z - z\| \to 0 \quad \text{when} \quad h \to 0, \quad \text{for any } z \in Z. \ \blacksquare$$

LEMMA 3.4. *Let* $\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$ *be a discretisation of* H. *If* $\mathfrak{A}$ *approximates* H *then*

$$\forall x \in H \quad \|\pi_h x - x\| \to 0 \quad \textit{when} \quad h \to 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary and let z_ε be an element of a dense subset $Z \subset H$ such that $\|x - z_\varepsilon\| < \varepsilon$, where x is an arbitrarily fixed element of H. Then

(2) Here V_h need not be finite-dimensional.

$$\|\pi_h x - x\| \leqslant \|\pi_h(x-z_\varepsilon)\| + \|\pi_h z_\varepsilon - z_\varepsilon\| + \|z_\varepsilon - x\| \leqslant K\cdot\varepsilon + \varepsilon + \|\pi_h z_\varepsilon - z_\varepsilon\|$$

because of equicontinuity of the family $\{\pi_h\}_{h\in\Theta}$.

Now we can choose h_0 such that for any $h \in \Theta$, $0 < h < h_0$, we get

$$\|\pi_h z_\varepsilon - z_\varepsilon\| < \varepsilon.$$

It follows that

$$\|\pi_h x - x\| \leqslant \varepsilon(2+K)$$

if $h < h_0$, $h \in \Theta$. ■

Adjoint discretisation of the space H

Let $\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$ be a discretisation of the space H. Let $e_1^h, \ldots, e_{d_h}^h$ be the basis of unit coordinate vectors in X_h. Put $\varphi_j^h = p_h e_j^h$, $j = 1, 2, \ldots, d_h$.

Then the projection $\pi_h = p_h r_h$

$$\pi_h\colon H \xrightarrow[\text{onto}]{} V_h = p_h(X_h) \subset H$$

is of the form given by (2.2):

$$\forall x \in H \qquad \pi_h x = \sum_{j=1}^{d_h} \varphi_j^h (x, \psi_j^h),$$

where the systems $\varphi_1^h, \ldots, \varphi_{d_h}^h$ and $\psi_1^h, \ldots, \psi_{d_h}^h$ are mutually biorthonormal, d_h being the dimension of X_h. From Lemma 2.4 we get

$$\forall x \in H \qquad \pi_h^* x = \sum_{j=1}^{d_h} \psi_j^h (x, \varphi_j^h) = q_h s_h x,$$

where:

$$\begin{aligned} &s_h\colon H \to X_h,\\ &s_h x = [(x, \varphi_1^h), \ldots, (x, \varphi_{d_h}^h)]^{\mathrm{T}}, \qquad x \in H,\\ (3.3)\qquad &q_h\colon X_h \to H,\\ &q_h x_h = \sum_{j=1}^{d_h} \psi_j^h x_j^h, \qquad x^h = [x_1^d, \ldots, x_{d_h}^h]^{\mathrm{T}} \in X_h \end{aligned}$$

are linear continuous maps (see Lemma 2.6).

Denote

$$\mathfrak{A}^* = \{X_h, q_h, s_h\}_{h\in\Theta}.$$

It is easy to verify that $\mathfrak{A}^*$ is also a discretisation of the space H. We call $\mathfrak{A}^*$ the *adjoint discretisation of H* (with respect to $\mathfrak{A}$).

The implications:

$$\mathfrak{A} \text{ approximates } H \Rightarrow \mathfrak{A}^* \text{ approximates } H;$$
$$\mathfrak{A} \text{ is stable} \Rightarrow \mathfrak{A}^* \text{ is stable}$$

in general fail.

4. Discretisation of compact operators

Consider a pair of Hilbert spaces V and H with the scalar products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$, respectively. Let T be a linear compact map,

$$T\colon V \to H.$$

Let

$$\mathfrak{A}_H = \{X_h^H, p_h^H, r_h^H\}_{h\in\Theta}; \qquad \pi_h^H = p_h^H r_h^H,$$
$$\mathfrak{A}_V = \{X_h^V, p_h^V, r_h^V\}_{h\in\Theta}; \qquad \pi_h^V = p_h^V, r_h^V,$$

be discretisations of the spaces H and V and let

$$\mathfrak{A}_H^* = \{X_h^H, q_h^H, s_h^H\}_{h\in\Theta}; \qquad \pi_h^{H*} = q_h^H s_h^H,$$
$$\mathfrak{A}_V^* = \{X_h^V, q_h^V, s_h^V\}_{h\in\Theta}; \qquad \pi_h^{V*} = q_h^V s_h^V$$

be their adjoints.

Define the maps

$$\hat{T}_h = r_h^H T q_h^V,$$
$$T_h = \pi_h^H T \pi_h^{V*} = p_h^H \hat{T}_h s_h^V;$$

then

$$\hat{T}_h\colon X_h^V \to X_h^H,$$
$$T_h\colon V \to H.$$

We call $\{\hat{T}_h\}_{h\in\Theta}$ the *discretisation of T induced by $\mathfrak{A}_H$ and $\mathfrak{A}_V$*. There are many possibilities of defining a discretisation of T; this one, however, seems to be rather convenient for our aims, because:

(a) in the case $V = H$ it involves only one basis, namely $\psi_1^h, \ldots, \psi_{d_h}^h$;

(b) the assumptions concerning $\mathfrak{A}_H$ and $\mathfrak{A}_V$ for the convergence properties of $\hat{T}_h$, in this case, seem to be rather natural (see Theorem 4.3).

LEMMA 4.1 (see [4]). *Let $T\colon V \to H$ be compact, and let $K_h\colon H \to H$, $h \in \Theta$, be a family of linear maps such that*

$$\forall x \in H \qquad \|x - K_h x\|_H \to 0 \quad \textit{when} \quad h \to 0.$$

Then

$$\|(I-K_h)T\|_{VH} \to 0 \quad \textit{when} \quad h \to 0. \tag{4.1}$$

Proof. Suppose (4.1) to be not true. Then

$$\exists q > 0\, \forall h_0\, \exists h \in \Theta,\, h < h_0 \qquad \|(I-K_h)T\|_{VH} > q > 0,$$

i.e. taking $h_0 = 1/k$, $k = 1, 2, \ldots$, we can find a sequence of points $x_k \in V$ and a sequence of corresponding numbers $h_k \in \Theta$, $0 < h_k < 1/k$, such that $\|x_k\|_V = 1$ and

$$\|(I-K_{h_k})Tx_k\|_H > q > 0. \tag{4.2}$$

Since T is compact and $\|x_k\|_V = 1$, without loss of generality we can assume that the sequence of points $y_k = Tx_k$ converges strongly to some element $y \in H$.

Hence

$$\|(I-K_{h_k})Tx_k\|_H \leqslant \|(I-K_{h_k})(y_k-y)\|_H+\|(I-K_{h_k})y\|_H$$
$$\leqslant \|I-K_{h_k}\|_H\|y_k-y\|_H+\|(I-K_{h_k})y\|_H.$$

Because of uniform boundedness of the family K_{h_k}, the first term tends to zero (Banach–Steinhaus Theorem). The second term converges to zero by the properties of K_h. This contradicts (4.2). ■

LEMMA 4.2 (see [4]). *Assume T: $V \to H$ to be compact and K_h: $V \to V$, $h \in \Theta$, to be a family of linear maps such that:*

(4.3) $$\forall x \in V \quad \|(I-K_h)x\|_V \to 0 \quad \text{when} \quad h \to 0.$$

Then

(4.4) $$\|T(I-K_h^*)\|_{VH} \to 0 \quad \text{when} \quad h \to 0.$$

Proof. Suppose (4.3) to be not true. Then, just as previously, we can find sequences $x_k \in V$, $h_k \in \Theta$, $k = 1, 2, \ldots$, satisfying

$$\|x_k\|_V = 1$$

and

(4.5) $$\|T(I-K_{h_k}^*)x_k\|_H > q > 0,$$

and such that the sequence

$$z_k = T(I-K_{h_k}^*)x_k \in H$$

tends strongly to some element $z \in H$. Hence

(4.6) $$(z_k, z)_H = \big(T(I-K_{h_k}^*)x_k, z\big)_H \to \|z\|_H^2 \quad \text{when} \quad k \to \infty.$$

But since T: $V \to H$ is continuous, T^*: $H \to V$ exists and

$$(z_k, z)_H = \big((I-K_{h_k}^*)x_k, T^*z\big)_V = \big(x_k, (I-K_{h_k})T^*z\big)_V$$
$$\leqslant \|x_k\|_V\|(I-K_{h_k})T^*z\|_V \to 0$$

when $k \to \infty$ in view of (4.3); i.e. $z = 0$, which contradicts (4.5). ■

THEOREM 4.3 (see also [4]). *Let T: $V \to H$ be a linear compact map and assume that $\mathfrak{A}_H$ and $\mathfrak{A}_V$ approximate the spaces H and V, respectively. Then*

$$\|T-T_h\|_{VH} \leqslant \|(I-\pi_h^H)T\|_{VH}+\|\pi_h^H\|_H\|T(I-\pi_h^{V^*})\|_{VH} \to 0 \quad \text{when} \quad h \to 0.$$

Proof. Observe that

$$T-T_h = (I-\pi_h^H)T+\pi_h^H T(I-\pi_h^{V^*}).$$

By the uniform boundedness of the family $\{\pi_h^H\}_{h\in\Theta}$ and by Lemmas 4.1 and 4.2 we get

$$\|T-T_h\|_{VH} \to 0 \quad \text{when} \quad h \to 0. \ ■$$

Now assume that $H = V$ and $\mathfrak{A}_V = \mathfrak{A}_H$. We have the following:

COROLLARY. *If $H = V$ and $\mathfrak{A}_H$ approximates H, then*

$$\|T-T_h\|_H \to 0.$$

5. Some general lemmas on eigenvalues

LEMMA 5.1. *Let B: $H \to H$ be a linear operator and let π: $H \to H$ be a linear projection, i.e. $\pi\pi = \pi$.*

Assume that

(5.1) $$B(I-\pi) = 0.$$

Then for an integer $l > 0$ and an element $u \in H$, $u \neq 0$, and for a complex number λ we have

$$[B-\lambda I]^l u = 0$$

iff

(5.2) $$[\pi B-\lambda I]^l y = 0$$

and

$$(I-\pi)B\sum_{j=1}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^j y+(-\lambda)^l z = 0,$$

where $y = \pi u$, $z = (I-\pi)u$.

Proof. We have

$$(B-\lambda I)^l = \sum_{j=0}^{l}\binom{l}{j}(-\lambda)^{l-j}B^j.$$

On the other hand,

$$B = \pi B+(I-\pi)B$$

and

$$B^j = \begin{cases}(\pi B)^j+(I-\pi)B(\pi B)^{j-1} & \text{for} \quad j>0,\\ I & \text{for} \quad j=0.\end{cases}$$

Hence

$$(B-\lambda I)^l = \sum_{j=0}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^j+(I-\pi)B\sum_{j=1}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^{j-1}$$

and for $u = y+z$

(5.3) $$[B-\lambda I]^l u = (\pi B-\lambda I)^l y+\sum_{j=0}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^j z+$$
$$+(I-\pi)B\sum_{j=1}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^{j-1}y+$$
$$+(I-\pi)B\sum_{j=1}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^{j-1}z$$
$$= (\pi B-\lambda I)^l y+(-\lambda)^l z+(I-\pi)B\sum_{j=1}^{l}\binom{l}{j}(-\lambda)^{l-j}(\pi B)^{j-1}y$$

because of (5.1).

Now observe that if $x_1 \in \pi H$ and $x_2 \in (I-\pi)H$ and $x_1 \neq 0$, $x_2 \neq 0$ then x_1 and x_2 are linearly independent.

Furthermore, observe that if $y \in \pi H$ and $z \in (I-\pi)H$, then

$$(\pi B-\lambda I)^l y = \sum_{j=0}^{l} \binom{l}{j} (\pi B)^j (-\lambda)^{l-j} y \in \pi H \tag{5.4}$$

and

$$(I-\pi)B \sum_{j=1}^{l} \binom{l}{j} (\pi B)^j (-\lambda)^{l-j} y + (-\lambda)^l z \in (I-\pi)H,$$

i.e. the terms

$$(\pi B-\lambda I)^l y \quad \text{and} \quad (I-\pi)B \sum_{j=1}^{l} \binom{l}{j} (\pi B)^j (-\lambda)^{l-j} y + (-\lambda)^l z \tag{5.5}$$

are linearly independent (if do not vanish).

Now if $(B-\lambda I)^l u = 0$, then using (5.3) we see that both terms (5.5) must be equal to zero. Conversely, if both terms (5.5) are zero, then from (5.3) it follows that

$$(B-\lambda I)^l u = 0. \blacksquare$$

LEMMA 5.2. *Let $\mathfrak{A} = \{X_h, p_h, r_h\}_{h\in\Theta}$ be an arbitrary discretisation of the space H and let*

$$G_h \colon X_h \to X_h$$

be a finite dimensional linear operator (a matrix).

Put

$$C_h = p_h G_h r_h.$$

Then

$$C_h \colon H \to H$$

and

(i) *the non-zero eigenvalues of G_h and C_h are equal and have the same algebraic multiplicities;*

(ii) *if $\lambda = 0$ is an eigenvalue of G_h then it is also an eigenvalue of C_h;*

(iii) *the (generalized) eigenfunctions corresponding to non-zero eigenvalues of C_h are contained in the subspace $V_h = p_h(X_h) \subset H$;*

(iv) *If $x_h \in X_h$ is a (generalized) eigenfunction corresponding to a non-zero eigenvalue of G_h then $x = p_h x_h$ is a (generalized) eigenfunction of C_h corresponding to the same eigenvalue. If x is a (generalized) eigenfunction of C_h corresponding to a non-zero eigenvalue, then $x = p_h x_h$, where x_h is (generalized) eigenfunction of G_h corresponding to the same eigenvalue.*

Proof. Observe that for any integer $k > 0$ and for any $x = p_h x_h$, $x_h \in X_h$, the following formula holds:

$$(C_h-\lambda I)^k x = p_h (G_h-\lambda I)^k x_h. \tag{5.6}$$

Hence if x_h is a (generalized) eigenfunction of G_h, then $x = p_h x_h$ is a (generalized) eigenfunction of C_h. The multiplicities of corresponding eigenvalues are equal.

If now

$$(C_h-\lambda I)^k x = 0, \quad x \neq 0,$$

with $\lambda \neq 0$, then:

$$(C_h-\lambda I)^k x = \sum_{p=0}^{k-1} (-\lambda)^p C_h^{k-p} \beta_p x + (-\lambda)^k x = 0, \quad \beta_p = \binom{k}{p},$$

i.e.

$$x = -\sum_{p=0}^{k-1} \left(\frac{C_h}{-\lambda}\right)^{k-p} \beta_p x \in C_h H \subset V_h = \pi_h H.$$

Hence $x = p_h x_h$, $x_h \in X_h$ (because $p_h \colon X_h \xrightarrow[\text{onto}]{} V_h \subset H$) and by (5.6), x_h is a generalized eigenfunction of G_h. $\blacksquare$

Remark. Assume that $H = V$ and $\mathfrak{A}_V = \mathfrak{A}_H$. Lemmas 5.1 and 5.2 reduce the eigenvalue problem for the operator T_h to the eigenvalue problem for the finite dimensional operator $\hat{T}_h$ (see § 4).

First observe that

$$T_h(I-\pi_h^{H^*}) = T_h - T_h \pi_h^{H^*} = T_h - \pi_h^H T \pi_h^{H^*} \pi_h^{H^*} = T_h - T_h = 0.$$

Hence, by Lemma 5.1, we can replace T_h by

$$\pi_h^{H^*} T_h = q_h^H s_h^H p_h^H \hat{T}_h^H s_h^H$$

and by Lemma 5.2 (for $\mathfrak{A}_H^*$ instead of $\mathfrak{A}_H$) we can replace this last operator by

$$s_h^H p_h^H \hat{T}_h \colon X_h^H \to X_h^H. \tag{5.7}$$

It is easy to see that

$$s_h^H p_h^H \hat{T}_h = G_{\varphi_h^H} \hat{T}_h, \tag{5.8}$$

where $G_{\varphi_h^H}$ is the Gramm matrix of the basis $\varphi_1^{H_h}, \ldots, \varphi_{j_h^H}^{H_h}$ of the discretisation $\mathfrak{A}_H$.

6. Discretisation of operators with compact inverse

Consider two Hilbert spaces V and H with scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$ respectively, and such that there exists a *compact* map J

$$J \colon V \to H \quad \text{(an imbedding)}.$$

Let

$$\mathfrak{A}_H = \{X_h^H, p_h^H, r_h^H\}_{h\in\Theta},$$
$$\mathfrak{A}_H^* = \{X_h^H, q_h^H, s_h^H\}_{h\in\Theta},$$
$$\mathfrak{A}_V = \{X_h^V, p_h^V, r_h^V\}_{h\in\Theta},$$
$$\mathfrak{A}_V^* = \{X_h^V, q_h^V, s_h^V\}_{h\in\Theta}$$

be discretisations of H and V.

Denote as before

$$\pi_h^H = p_h^H r_h^H, \qquad \pi_h^{H^*} = q_h^H s_h^H;$$
$$\pi_h^V = p_h^V r_h^V, \qquad \pi_h^V = q_h^V s_h^V.$$

Consider a linear continuous map

(6.1) $$A\colon V \to H$$

and put

$$\hat{A}_h = s_h^H A p_h^V,$$
$$A_h = q_h^H \hat{A}_h r_h^V = \pi_h^{H^*} A \pi_h^V.$$

Then

$$\hat{A}_h\colon X_h^V \to X_h^H,$$
$$A_h\colon V \to H.$$

We call $\{\hat{A}_h\}_{h\in\Theta}$ the *discretisation of the operator A induced by* $\mathfrak{A}_H$ *and* $\mathfrak{A}_V$; $\hat{A}_h$ is a $(d_h^H \times d_h^V)$-matrix.

Observe that the definition of $\hat{A}_h$ involves only the bases $\varphi_1^{Hh}, \ldots, \varphi_{d_h^H}^{Hh}$ and $\varphi_1^{Vh}, \ldots, \varphi_{d_h^H}^{Vh}$.

In principle, it is possible to define the discretisation of A in another way (see for instance § 4), but this definition seems to be convenient for our investigations.

A discretisation $\{\hat{A}_h\}_{h\in\Theta}$ of the operator A is called *stable* iff there exist $h_0 > 0$ and a positive constant C such that for any $h \in \Theta$, $h \leqslant h_0$,

(6.2) $$\hat{A}_h^{-1}\colon X_h^H \to X_h^V \text{ exists}$$

and

$$\|\hat{A}_h^{-1}\|_{X_h^H X_h^V} \leqslant C.$$

(In this case $d_h^H = d_h^V$!)

THEOREM 6.1. *Let the topology in* X_h^H *and* X_h^V *be defined by* (2.14). *Assume that*:

(i) *the discretisation* $\mathfrak{A}_H = \{X_h^H, p_h^H, r_h^H\}_{h\in\Theta}$ *is stable and approximates the space* H;

(ii) *the discretisation* $\mathfrak{A}_V = \{X_h^V, p_h^V, r_h^V\}_{h\in\Theta}$ *is stable*;

(iii) $d_h^H = d_h^V$;

(iv) *the operator* $A\colon V\to H$ *is bounded and coercive on* V, *i.e.*

(6.3) $$\|A\|_{VH} < \infty \quad \text{and} \quad \exists \gamma > 0\ \forall x \in V \quad (Ax, Jx)_H \geqslant \gamma \|x\|_V^2.$$

Then the discretisation $\{\hat{A}_h\}_{h\in\Theta}$ *is stable.*

Proof. Put

$$J_h = r_h^H J p_h^V.$$

Then

$$J_h\colon X_h^V \to X_h^H$$

is a family of uniformly bounded maps: $\|J_h\|_{X_h^V X_h^H} \leqslant \alpha$ with α not depending on $h \in \Theta$, $h < h_0$.

Using (2.14) and (6.3) we get for any $v \in X_h^V$

$$(\hat{A}_h v, J_h v)_{X_h^H} = (q_h^H \hat{A}_h v, p_h^H J_h v)_H = (\pi_h^{H^*} A p_h^V v, \pi_h^H J p_h^V v)_H$$
$$= (A p_h^V v, \pi_h^H J p_h^V v)_H = (A p_h^V v, J p_h^V v)_H - (A p_h^V v, (I-\pi_h^H) J p_h^V v)_H$$
$$\geqslant \gamma \|p_h^V v\|_V^2 - \|A\|_{VH} \|(I-\pi_h^H) J\|_{VH} \|p_h^V v\|_V^2.$$

By the compactness of J and by (i), using Lemma 4.1, we get

$$(\hat{A}_h v, J_h v)_{X_h^H} \geqslant \gamma_1 \|p_h^V v\|_V^2$$

with the constant γ_1 not depending on $h \in \Theta$, $h < h_0$. Now, using Lemma 3.1, we verify that

$$\|p_h^V v\|_V \geqslant \beta \|v\|_{X_h^V}$$

for β not depending on $h \in \Theta$, $h < h_0$. Finally,

$$\|\hat{A}_h v\|_{X_h^H} \alpha \|v\|_{X_h^V} \geqslant (\hat{A}_h v, J_h v)_{X_h^H} \geqslant K_1 \|v\|_{X_h^V}^2$$

or

$$\|A_h v\|_{X_h^H} \geqslant K \|v\|_{X_h^V},$$

where K is some constant not depending on $h \in \Theta$, $h < h_0$. From this last inequality follows the existence of $\hat{A}_h^{-1}$, which is uniformly bounded. Since $\hat{A}_h$ is defined over the whole space X_h^V, $d_h^V = d_h^H$, and $\hat{A}_h$ is one-to-one, it has to be onto the whole space X_h^H. In other words,

$$\hat{A}_h^{-1}\colon X_h^H \to X_h^V, \qquad h \in \Theta,\ h < h_0. \ \blacksquare$$

Remark. From inequality (6.3) it follows easily that in this case the map J must be one-to-one. ■

Now put

$$B_h = p_h^V \hat{A}_h^{-1} s_h^H.$$

It is clear that if the families $\{p_h^V\}_{h\in\Theta}$, $\{s_h^H\}_{h\in\Theta}$ are uniformly bounded and the discretisation $\{\hat{A}_h\}_{h\in\Theta}$ is stable, then the operators

$$B_h\colon H \to V$$

are uniformly bounded, i.e.

$$\|B_h\|_{HV} < K$$

for $h \in \Theta$, $h < h_0$, with K not depending on h.

Observe that

(6.4) $$\pi_h^V B_h = p_h^V r_h^V p_h^V \hat{A}_h^{-1} s_h^H = p_h^V \hat{A}_h^{-1} s_h^H = B_h.$$

Assume that the linear continuous map A

$$A\colon V \xrightarrow[\text{onto}]{} H$$

has the *continuous inverse* $A^{-1} = T$

$$T\colon H \xrightarrow[\text{onto}]{} V.$$

Put

$$S = JT,$$
$$S\colon H \to H;$$

then S is compact.

Denote

$$S_h = \pi_h^H S \pi_h^{H^*}$$

(see § 4). We have the following

THEOREM 6.2. *Assume that:*

(i) *for some $h_0 > 0$ the families of maps*

$$\{s_h^H\}_{\substack{h\in\Theta,\\ h<h_0}}, \quad \{p_h^V\}_{\substack{h\in\Theta\\ h<h_0}}$$

are uniformly bounded;

(ii) $\mathfrak{A}_H$ *approximates* H;

(iii) $\{\hat{A}_h\}_{h\in\Theta}$ *is stable.*

Then there exists a constant C not depending on $h\in\Theta$ such that

$$\|JB_h - S\|_H \leqslant C\|S-S_h\|_H \to 0 \quad \text{when} \quad h\to 0.$$

Proof. We have

$$\begin{aligned}(S-S_h)AB_h &= SAB_h - S_hAB_h = JA^{-1}AB_h - S_hAB_h\\ &= JB_h - \pi_h^H Sq_h^H s_h^H Ap_h^V \hat{A}_h^{-1} s_h^H = JB_h - \pi_h^H Sq_h^H \hat{A}_h \hat{A}_h^{-1} s_h^H\\ &= JB_h - S_h = JB_h - S + S - S_h.\end{aligned}$$

Hence

$$JB_h - S = (S-S_h)AB_h - (S-S_h)$$

and

$$\|JB_h - S\|_H \leqslant [\|A\|_{VH}\|B_h\|_{HV}+1]\|S-S_h\|_H \leqslant [\|A\|_{VH}\cdot K+1]\|S-S_h\|_H,$$

where

$$\|B_h\|_{HV} \leqslant K.$$

By Theorem 4.3 we get

$$\|JB_h - S_h\|_H \leqslant C\|S-S_h\|_H \to 0 \quad \text{when} \quad h\to 0. \;\blacksquare$$

DEFINITION. We call the *spectrum of the pair of operators A, J* the set of all numbers $\lambda = 1/\mu$ where μ is in the spectrum of the compact operator JA^{-1},

$$JA^{-1}\colon H\to H$$

(see [5]).

Using this definition we can state the following

THEOREM 6.3. *Assume the conditions of Theorem 6.2. Let G_h be the Gramm matrix of the form:*

$$G_h = (g_{ij}), \quad g_{ij} = (J\varphi_j^{Vh}, \varphi_i^{Hh})_H, \quad i,j = 1,2,\ldots,d_h.$$

Then the spectrum of the following generalized matrix-eigenvalue problem:

$$[\hat{A}_h - \lambda G_h]x = 0 \tag{6.5}$$

approximates the spectrum of the operator A in the following sense:

Consider the eigenvalues of the operator A lying inside a fixed arbitrary circle C centred at zero on the complex plane, and the collection of ε-discs centred at these eigenvalues, for any $\varepsilon > 0$.

For every $\varepsilon > 0$ there exists $h_{\varepsilon C}$ satisfying the condition: If $h < h_{\varepsilon C}$ and $h\in\Theta$, then the above ε-discs contain the number of eigenvalues of (6.5) equal to multiplicities of the corresponding eigenvalues of A (counting their multiplicities). The other eigenvalues of (6.5), if exist, lie outside the circle C.

Proof. From the stability of $\{\hat{A}_h\}_{h\in\Theta}$ it follows that $d_h = d_h^H$, i.e. $\hat{A}_h$ and G_h in (6.5) are square matrices. Observe that

$$JB_h(I-\pi_h^{H^*}) = JB_h - JB_h\pi_h^{H^*} = JB_h - Jp_h^V\hat{A}_h^{-1}s_h^H q_h^H s_h^H = JB_h - JB_h = 0$$

because $s_h^H q_h^H = I$. Hence we can apply Lemma 5.1 and replace the operator JB_h by $\pi_h^{H^*}JB_h = q_h^H s_h^H Jp_h^V \hat{A}_h^{-1} s_h^H$.

Now using Lemma 5.2 we replace the above operator by the matrix

$$s_h^H Jp_h^V \hat{A}_h^{-1} = G_h\hat{A}_h^{-1},$$

$\hat{A}_h^{-1}$ being uniformly bounded for $h\in\Theta$, $h<h_0$. Now, it is enough to apply Theorem 6.2 and the Perturbation Theorem. ■

Observe that the matrix eigenvalue problem (6.5), approximating the original eigenvalue problem for the operator A, depends only on the bases:

$$\{\varphi_1^{Hh}, \ldots, \varphi_{d_h}^{Hh}\} \quad \text{and} \quad \{\varphi_1^{Vh}, \ldots, \varphi_{d_h}^{Vh}\} \tag{6.6}$$

(see (2.15) and (2.18)).

Only the assumption (ii) in Theorems 6.2 and 6.3 makes use directly of the approximating property of discretisation $\mathfrak{A}_H$, that is, involves both bases $\{\varphi_1^{Hh}, \ldots, \varphi_{d_h}^{Hh}\}$ and $\{\psi_1^{Hh}, \ldots, \psi_{d_h}^{Hh}\}$.

We can, however, omit entirely the use of bases other than (6.6). We have the following

THEOREM 6.4. *Assume that:*

(i) *For some $h_0 > 0$ the families of maps*

$$\{s_h^H\}_{\substack{h\in\Theta,\\ h<h_0}}, \quad \{p_h^V\}_{\substack{h\in\Theta\\ h<h_0}}$$

are uniformly bounded;

(ii) *For the subspaces V_h of H, spanned by the bases $\{\varphi_1^{Hh}, \ldots, \varphi_{d_h}^{Hh}\}$, the family $\{V_h\}_{h\in\Theta}$ approximates the space H* (see § 3);

(iii) *The discretisation $\{A_h\}_{h\in\Theta}$ is stable.*

Then the matrix eigenvalue problem (6.5) approximates the original eigenvalue problem for the operator A in the sense of Theorem 6.3.

Proof. Let $\mathfrak{A}_H$ be the discretisation of the space H defined by a family of orthogonal projections $\{\pi_h^H\}_{h\in\Theta}$ onto subspaces V_h. Since the family $\{\pi_h^H\}_{h\in\Theta}$ is uniformly bounded, then from Lemma 3.2 it follows that $\mathfrak{A}_H$ approximates H. By Lemma 2.5 we see that π_h^H is entirely determined by the basis $\{\varphi_1^{Hh}, \ldots, \varphi_{d_h}^{Hh}\}$. It now suffices to apply Theorem 6.3. ■

The trivial example of a problem satisfying conditions (6.1) and (6.2) stated in this paragraph is the following

$$\Delta u = \lambda u \quad \text{on } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^d with regular boundary. In this case we can take $V = H_0^2(\Omega)$, $H = H^0(\Omega) = L_2(\Omega)$ and $A = \Delta$ (in generalized sense). In this case, by Rellich Theorem, the imbedding $J: V \to H$ is a compact linear map, and A and A^{-1} are bounded operators.

More advanced examples of elliptic problems are discussed also in [5] (see Section 4, as well as works quoted in this paper).

Remark. In the paper [5] there are given error bounds obtained in a similar way, in the case of a differential operator in the Sobolev space. The information about the rate of convergence is obtained when the spaces $V_h = p_h(X_h)$ are special finite-dimensional subspaces of piecewise polynomials.

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ОБ ЭФФЕКТИВНЫХ МЕТОДАХ ВЫЧИСЛЕНИЯ СОБСТВЕННЫХ ЧИСЕЛ И СОБСТВЕННЫХ ВЕКТОРОВ С ПОМОЩЬЮ МЕТОДА ЯКОБИ И МЕТОДА ЛОЖНЫХ ВОЗМУЩЕНИЙ

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В заметке содержится часть лекций, прочитанных автором во время работы семестра ,,Математические модели и численные методы'' Международного математического центра им. Стефана Банаха в Варшаве. Описываются эффективные методы приближенного вычисления отдельного собственного числа и соответствующего ему собственного вектора симметрической матрицы, состоящие из двух этапов. На первом этапе применяется метод Якоби элементарных плоских вращений до тех пор, пока мы не сможем указать отрезок на вещественной оси, содержащий интересующее нас собственное число. На втором этапе применяется метод ложных возмущений для уточнения собственного числа и собственного вектора.

Метод Якоби, являющийся ,,глобальным'' методом для полной проблемы собственных чисел, устанавливает при этом условия, достаточные для применимости и быстрой сходимости метода ложных возмущений. Метод ложных возмущений является ,,локальным'' методом для частной проблемы собственных чисел.

Идея соединения этих двух методов содержится в работе [12] (см. также [13]).

1. Теоремы теории возмущения спектра

Приведем некоторые основные теоремы теории возмущения спектра самосопряженных операторов A в гильбертовом пространстве H со скалярным произведением $(\cdot,\cdot)$ (детальное изложение приведенных здесь фактов можно найти, например, в монографиях [1] и [8], см. также [9]).

Теорема 1. *Пусть A самосопряженный оператор в гильбертовом пространстве H, B ограниченный самосопряженный оператор и пусть E_t и E'_t обозначают*