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BASIC CONCEPTS IN NUMERICAL ERROR ANALYSIS (SUGGESTED BY LINEAR ALGEBRA PROBLEMS)

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1. Numerical arithmetic (fl)

We consider computations performed in floating-point arithmetic (fl). Assuming that neither over- nor under-flow phenomena occur, we characterize an fl-arithmetic by a single, small positive number ϱ: $\mathrm{fl}\langle\varrho\rangle$. We assume that the reader is familiar with the rules of number (or vector, or matrix) representation in $\mathrm{fl}\langle\varrho\rangle$ and with the rules of execution of the arithmetic fl-operations (cf. Wilkinson [7]).

Comments

1. In the case of the standard binary fl-arithmetic with t digits mantissa, ϱ equals 2^{-t}. Typically $\varrho \in [10^{-16}, 10^{-6}]$.

2. One can define the representation of number in $\mathrm{fl}\langle\varrho\rangle$ introducing the "rounding function" $\mathrm{rd}_\varrho\colon X \to X_\varrho \subset X \subset R$, R being a space of real or complex numbers (cf. Stoer [5]). The main properties of rd_ϱ and X_ϱ are:

$$\forall_{x\in X} |x-\mathrm{rd}_\varrho(x)| \leqslant \varrho|x|, \quad X_\varrho = \{x \in X; x = \mathrm{rd}_\varrho(x)\}.$$

Typically

$$X = \{x \in R; x = 0 \vee \eta < |x| < \eta^{-1}\}, \quad 10^{-160} < \eta < 10^{-40}.$$

3. Denoting by ∇ an arbitrary arithmetic operator: $+, -, \times, /$, we can assume for real operations in $\mathrm{fl}\langle\varrho\rangle$ with well-constructed arithmetic unit the following: if $x, y \in X_\varrho$ and $z = x\nabla y \in X$ then the computed result, z_ϱ, equals $\mathrm{rd}_\varrho(z)$.

For complex operations we obtain in this case the relation:

$$|z_\varrho - (x\nabla y)| \leqslant \varrho \cdot k \cdot |z_\varrho|,$$

where k depends only on the type of operation (in most cases $1 \leqslant k \leqslant 6$).

2. A simple model of a numerical problem

We consider two normed cartesian spaces, the *input-data-space* R_d and the *result-space* R_r. Let A_0 be an open subset of R_d and let us consider a mapping φ: $A_0 \to R_r$.

The problem of an effective construction of the element $\varphi(a)$ for a given $a \in A_0$ is denoted by $\{\varphi, a\}$. We shall call it a *single-data problem*. The problem of the effective construction of $\varphi(a)$ for any $a \in A \subset A_0$ is denoted by $\{\varphi, A\}$ (a *set-of-data problem*).

We assume further that if $\check{a}$ is the representation of $a \in A_0$ in $\mathrm{fl}\langle\varrho\rangle$ then the inequality $\|a-\check{a}\| \leqslant \varrho_d \cdot \|a\|$ holds, where $\varrho_d = \varrho \cdot k_d$, k_d being a positive number which depends on A_0, the norm and dimension of R_d. We assume, similarly, the relation

$$\|b-\check{b}\| \leqslant \varrho_r \cdot \|b\| \quad \text{for} \quad b \in \varphi(A_0).$$

Comments

1. By a "cartesian space" we understand a finite-dimensional space, given as the cartesian product $R \times R \times \ldots \times R$, where R is the space of real or complex numbers. The usual linear operations and usual norm postulates for this space are assumed.

2. The concept of an "effective construction" can be more precisely explained by referring to the list of some elementary admissible operations, which in the case of numerical computation are essentially arithmetic operations.

3. If R_d is a vector space with any Schur norm, then $\varrho_d = \varrho$. The same holds if R_d is a matrix space with 1, ∞ or E-norm. If we consider the spectral norm (2-norm) then for the set $A_0 = R_d$ the equality $\varrho_d = \varrho \cdot \sqrt{m}$ holds, $(m \times m)$ being the dimension of R_d. Defining A_0 as the subset of nonnegative matrices in R_d, we have the equality $\varrho_d = \varrho$ for the spectral norm, too.

3. Finite-method algorithm

Let V be an algorithm which gives for any $a \in A \subset A_0$ in a finite number of elementary operations the element $x = \varphi(a) \in R_r$. We now consider the accurate execution of each operation; therefore the computed result, $V[a]$, equals $\varphi(a)$.

We define the algorithm V_ϱ which differs from V in the following:

(i) all numbers involved in V as data are replaced by their representations in $\mathrm{fl}\langle\varrho\rangle$,

(ii) all operations in V are replaced by the corresponding $\mathrm{fl}\langle\varrho\rangle$ operations.

We assume that for each $a \in A$ and for each sufficiently small ϱ all operations of the algorithm V_ϱ are well defined, i.e. the element $x_\varrho = V_\varrho[a] \in R_r$ is uniquely defined for ϱ sufficiently small. The element x_ϱ will be referred to as the *solution to the problem* $\{\varphi, a\}$, computed in $\mathrm{fl}\langle\varrho\rangle$ by means of the algorithm V.

Comments

1. We use square brackets to underline the difference between the algorithmic relation $x = V[a]$ and the functional relation $x = \varphi(a)$. Note that the expressions $V_1[a, b, c] = a \times (b+c)$ and $V_2[a, b, c] = a \times b + a \times c$ define in Algol 60 two different algorithms for the evaluation of the same function.

2. The data involved in the algorithm V consist mainly of input data of the problem, but may also contain some "own" data of the algorithm. For example, in the problem {compute the area of the circle with given radius a} the algorithm V defined by means of the expression $V[a] = a^2\pi$ uses as "own" data the number π. Some other algorithms need not use π as data.

4. Infinite-method algorithms

An infinite sequence of mappings $\{\varphi^{(p)}\}_{p=1}^{\infty}$, $\varphi^{(p)}$: $A_0 \to R_r$, such that for each $a \in A_0$ $\varphi^{(p)}(a) \to \varphi(a)$ when $p \to \infty$ will be referred to as an *infinite method* of approximating the mapping φ on A_0.

We assume that for each p, a finite-method algorithm $V^{(p)}$ solving the problem $\{\varphi^{(p)}, A\}$, $A \subset A_0$ is known. We assume that each algorithm $V^{(p)}$ defines uniquely for each $a \in A$ and for sufficiently small ϱ the element $x_\varrho^{(p)} = V_\varrho^{(p)}[a] \in R_r$. We assume further that for each $a \in A$ and for each sufficiently small ϱ there exists a positive integer $s = s(a, \varrho)$ such that the element $x_\varrho = x_\varrho^{(s)}$ can be considered as the solution to the problem $\{\varphi, a\}$, computed in $\mathrm{fl}\langle\varrho\rangle$ by means of algorithms $\{V^{(p)}\}$.

Comments

1. Practical methods of determining the index $s = s(a, \varrho)$ are not discussed here. Some of such methods are known as "termination criteria".

5. Well-behaving algorithms (WB)

We call an algorithm V (or a sequence $\{V^{(p)}\}$) *numerically well behaved* in a set-of-data problem $\{\varphi, A\}$, $A \subset A_0$, if there exist numbers K_d, K_r such that for each $a \in A$ and for each sufficiently small ϱ there exists $a_\varrho \in A_0$ such that the relations

$$\|a - a_\varrho\| \leqslant \varrho_d \cdot \|a\| \cdot K_d, \qquad \|\varphi(a_\varrho) - x_\varrho\| \leqslant \varrho_r \cdot \|\varphi(a_\varrho)\| \cdot K_r$$

hold.

Comments

1. The WB property of an algorithm is invariant with respect to the norms considered. However, a change of norms can influence the quantities K_d, K_r, ϱ_d, ϱ_r.

2. A WB-algorithm guarantees (provided sufficiently small ϱ is used) to produce for each single-data-problem $\{\varphi, a\}$, $a \in A$, "a slightly wrong solution to a slightly wrong problem", according to Kahan's postulate, cf. Kahan [4].

3. It also guarantees that the computed solution lies close to the "natural domain" caused by data representation errors, cf. Faddeev, Faddeeva [3].

4. The numbers K_d, K_r will be referred to as *error-accumulation-esimates* (EC-estimates) of the algorithm V for $\{\varphi, A\}$, provided the inequalities in WB definitions are sharp. The "asymptotically-statistical" EC-estimates (cf. Voevodin [6]) are EC-estimates for some subset $A_\sigma \subset A$ such that $\mathrm{mes}(A - A_\sigma) \ll \mathrm{mes}(A)$.

5. If φ is continuous in A_0 then for each $a \in A$ we have $x_\varrho \to \varphi(a)$ whenever $\varrho \to 0$.

6. The sequence of logical quantifiers in the WB definition is:

$$\exists K_d, K_r \,\forall a \in A \,\exists \varrho_0 > 0 \,\forall \varrho < \varrho_0 \,\exists a_\varrho \in A_0.$$

6. The optimal error-level of the solution

We now assume that for each element $a \in A_0$ the mapping φ fulfils the local Hölder condition in the neighbourhood of a.

This means that for each $a \in A_0$ there exist positive numbers d_0, α, H such that for each $\hat{a} \in A_0$, if $d = \|a-\hat{a}\| \leqslant d_0$ then $\|\varphi(a)-\varphi(\hat{a})\| \leqslant H \cdot d^\alpha$.

The choice of α and H depends to some extent on d_0.

To achieve the practical uniqueness of α and H we assume that for each $a \in A_0$ it is possible to choose such $d_0 = d_0(a) > 0$ that further decreasing of d_0 can neither significantly decrease $H = H(a)$ nor increase $\alpha = \alpha(a)$ in the above relations. The above mentioned class of mappings will be referred to as the $\varkappa$-class.

The sub-class of mappings fulfiling the local Lipschitz condition ($\alpha = 1$) will be referred to as the λ-class.

We define the *optimal-error-level* of the solution to the problem $\{\varphi, a\}$ in $\mathrm{fl}\langle\varrho\rangle$ by the expression:

$$P(\varphi; a, \varrho) = H(a)(\varrho_d \cdot \|a\|)^{\alpha(a)} + \varrho_r \cdot \|\varphi(a)\|.$$

Comments

1. In order to get some bounds for the error of the solution caused by the representation errors in the input data, we must assume some regularity of the mapping φ. The $\varkappa$-class seems to include all mappings involved in the algebraic problems.

2. $P(\varphi; a, \varrho)$ gives a bound for the sensitivity of the solution $x = \varphi(a)$ with respect to the representation errors in the input-data and in the solution itself.

3. The $\varkappa$-class of mappings is invariant with respect to the norms considered. However, such a replacement of norms influences eventually the quantities $d_0(a)$, $H(a)$, though the exponent $\alpha(a)$ is invariant for given φ and a.

7. The numerical stability of an algorithm (NS)

An algorithm V (or a sequence $\{V^{(s)}\}$) is said to be *numerically stable* in a set-of-data problem $\{\varphi, A\}$, if there exists a number K such that for each $a \in A$ and for each sufficiently small ϱ the relation

$$\|\varphi(a) - x_\varrho\| \leqslant K \cdot P(\varphi; a, \varrho)$$

holds where $x_\varrho = V_\varrho[a]$ or $x_\varrho = x_\varrho^{(s)} = V_\varrho^{(s)}[a]$.

Comments

1. The NS-property of an algorithm is invariant with respect to the norm considered. However, the number K depends on the specific choice of norms.

2. The main idea of the NS-concept is the comparison of the error of the computed solution (caused by the combined influence of the rounding errors in the algorithm and the possible truncation error of the infinite method) with the maximal perturbation of the solution. This maximal perturbation is caused by perturbations of the input data and the result itself.

The latter are of such order of magnitude as the corresponding representation errors. This idea was expressed among others by Bauer [2].

3. The number K has a similar meaning as EC-estimates in the WB-concept. It characterizes the error accumulating property of the algorithm for the considered set-of-data problem. The quantity

$$K^* = \sup_{a \in A} \overline{\lim_{\varrho \to 0}} \|\varphi(a) - x_\varrho\| / P(\varphi; a, \varrho)$$

is closely related to the quantity $\omega(a, b, L, P)$ proposed by Babuška ([1], (2.10)) in his definition of the NS-concept.

4. The following, almost trivial, theorem holds: if φ is an $\varkappa$-mapping on A_0 then any WB-algorithm in $\{\varphi, A\}$ is NS in $\{\varphi, A\}$ with $K = \max(K_d, K_r)$ whenever $\alpha^* = \max \alpha(a) \geqslant 1$, or $K = K_d$ whenever $\alpha^* < 1$.

8. The relative error bound and the condition of the problem

Let us assume $x = \varphi(a) \neq 0$ and let x_ϱ be the solution computed by a NS algorithm. Then the relation

$$\frac{\|x - x_\varrho\|}{\|x\|} \leqslant \left(C(a) \cdot \varrho_d^{\alpha(a)} + \varrho_r\right) \cdot K$$

holds. The quantity

$$C(a) = H(a) \frac{\|a\|^{\alpha(a)}}{\|x\|}$$

will be referred to as the *condition number* of the problem $\{\varphi, a\}$. The quantity $\alpha(a)$ will be referred to as the *condition exponent* of this problem.

Comments

1. Both $C(a)$ and $\alpha(a)$ characterize the relative sensitivity of the solution $\varphi(a)$ to small relative perturbations in the input data:

$$\frac{\|\varphi(a) - \varphi(\hat{a})\|}{\|\varphi(a)\|} \leqslant C(a) \cdot \left(\frac{\|a - \hat{a}\|}{\|a\|}\right)^{\alpha(a)},$$

$C(a)$ depends on the specific choice of norms in R_d, R_r.

2. For the Lipschitz case ($\alpha = 1$) the bound for the relative error can be written in the form:

$$\frac{\|x-x_\varrho\|}{\|x\|} \leqslant \big(C(a)+1\big)\cdot \max(\varrho_d, \varrho_r)\cdot K,$$

which exposes the role of all essential factors: the sensitivity of the problem, the accuracy of the numerical arithmetic, the error-accumulating properties of the algorithm.

3. For some problems and some WB-algorithms each element a_ϱ in the WB definition fulfils the relation

$$\|\varphi(a)-\varphi(a_\varrho)\| \leqslant H(a)\cdot \|a-a_\varrho\|^{\alpha(a)}.$$

We can refer to this (cf. Wilkinson [7]) as to the *specific correlation* of the perturbation $(a-a_\varrho)$ with the mapping φ. In such a case we obtain much greater accuracy in the computed solution than it could be expected regarding the condition of the problem. In fact, the problem is much better conditioned with respect to such specific perturbations in input data.

9. Superposition of algorithms

Let us consider a normed cartesian space R_p and mappings

$$\varphi_1\colon A_0 \to B_0 \subset R_p\ (A_0 \subset R_d), \qquad \varphi_2\colon B_0 \to R_w$$

such that $\varphi(a) = \varphi_2(\varphi_1(a))$ for each $a \in A_0$. Let V_1 be an algorithm solving the problem $\{\varphi_1, A\}$, $A \subset A_0$, V_2 an algorithm solving the problem $\{\varphi_2, B\}$, $B \subset B_0$, and let us assume that for each $a \in A$ for sufficiently small ϱ the element $V_{1\varrho}[a]$ is in B. We consider the algorithm V solving the $\{\varphi, A\}$ problem, defined by the relation $V[a] = V_2[V_1[a]]$.

The following theorems hold:

1. *If φ_2 is a λ-mapping on B_0 and the corresponding condition number is bounded: $C_2(b) \leqslant C_2$, $b \in B_0$, and if the algorithms V_1, V_2 are WB in $\{\varphi_1, A\}$, $\{\varphi_2, B\}$ with EC-estimates $K_d^{(i)}$, $K_r^{(i)}$, $i = 1, 2$, respectively, then the algorithm V is WB in $\{\varphi, A\}$ with EC-estimates*

$$K_d = K_d^{(1)}, \qquad K_r = K_r^{(2)}+(K_r^{(1)}+K_d^{(2)})\cdot C_2\cdot \varrho_p/\varrho_r.$$

2. *Let φ_1, φ_2 be λ-mappings. Assume that the condition for φ_2 is bounded: $C_2(b) \leqslant C_2$, $b \in B_0$, and that the quotient $\big(H_1(a)\cdot H_2(\varphi_1(a))\big)/H(a) \leqslant G$, $a \in A_0$, is bounded (H_i is the Lipschitz constant for φ_i, H for φ). If the algorithms V_1, V_2, are NS in $\{\varphi_1, A\}$, $\{\varphi_2, B\}$ with EC-estimates $K^{(1)}$, $K^{(2)}$, respectively, then the algorithm V is NS in $\{\varphi, A\}$ with the EC-estimate*

$$K \doteq \max\big(K^{(1)}G,\ K^{(2)}+(K^{(1)}+K^{(2)})\cdot C_2\cdot \varrho_p/\varrho_r\big).$$

Comment

For a specific class of problems (including typical linear algebra problems) a small perturbation in the solution can be always interpreted as a small perturbation in the input data. If $\{\varphi_1, A\}$ belongs to this class then the superposition of WB-algorithm is a WB-algorithm in $\{\varphi, A\}$ without any further assumptions.

10. Equations and residual stability (RS)

A numerical problem can be posed in the form of an equation. Let us consider a mapping $f\colon \mathcal{M} \subset R_d\times R_r \to R_e$, where R_e is linear, normed cartesian space. We assume that for each $a \in A_0$ the pair $(a, \varphi(a))$ belongs to $\mathcal{M}$ and $x = \varphi(a)$ is the unique solution of the equation $f(a, x) = 0$ in some neighbourhood of this point. We will refer to the element $r = f(a, \tilde{x}) \in R_e$ as the *residual element of the approximate solution* $\tilde{x}$ with respect to the equation $f(a, x) = 0$.

Let us assume that f fulfils in $\mathcal{M}$ the local Hölder condition with respect to a and x with coefficients $H_a(a, x)$, $H_x(a, x)$ and exponents $\alpha_a(a, x)$, $\alpha_x(a, x)$, respectively. We define the *optimal residual level of the equation* $f(a, x) = 0$ by the following expression:

$$Q(a, x, f, \varrho) = H_a(a, x)\cdot(\|a\|\cdot\varrho_d)^{\alpha_a(a,x)}+H_x(a, x)\cdot(\|x\|\cdot\varrho_r)^{\alpha_x(a,x)}.$$

The algorithm V is said to be *residually stable* for the problem $\{\varphi, A\}$ with respect to the equation $f(a, x) = 0$ if there exists a number K_e such that for each $a \in A$ and for sufficiently small ϱ the relation

$$\|f(a, x_\varrho)\| \leqslant K_e\cdot Q(a, x_\varrho, f, \varrho)$$

holds ($x_\varrho = V_\varrho[a]$ or $x_\varrho = V_\varrho^{(s)}[a]$).

Comments

1. The NS-concept is identical with the RS-concept with respect to the equation $f(a, x) = x-\varphi(a) = 0$.

2. An algorithm which is WB in $\{\varphi, A\}$ is RS with respect to any equation for which the quantity $\min\big(\alpha_a(a, \varphi(a)),\ \alpha_x(a, \varphi(a))\big)$ is bounded in A. We know some non trivial examples of equations such that the RS-property implies the NS-property or even the WB-property.

11. A general model of a numerical problem

We consider both the input data space R_d and the result space R_r in the form of the cartesian products of cartesian spaces:

$$R_d = R_{d1}\times R_{d2}\times \ldots \times R_{dl}; \qquad R_r = R_{r1}\times R_{r2}\times \ldots \times R_{rm}.$$

Provided the spaces R_{di}, R_{rj} are normed, we shall refer to this splitting as to the *structure of the spaces* R_d, R_r (shortly: $\Sigma(R_d, R_r)$). We denote the corresponding components of the elements of R_d or R_r, and consequently the components of mappings or algorithms, by adding the indices. Thus for example $a_i \in R_{di}$, $x_j^{(p)} = \varphi_j^{(p)}(a) \in R_{rj}$, $x_{\varrho j} = V_{\varrho j}[a] \in R_{rj}$ are the components of a, $x^{(p)} = \varphi^{(p)}(a)$, $x_\varrho = V_\varrho[a]$ respectively. The quantities ϱ_{di}, ϱ_{rj} denote the corresponding $\mathrm{fl}\langle\varrho\rangle$ characteristic numbers for the spaces R_{di}, R_{rj}.

11.1. WB-algorithms. An algorithm V (or $\{V^{(p)}\}$) is said to be WB in $\{\varphi, A\}$, $A \subset A_0$, with respect to a structure $\Sigma(R_d, R_r)$ if there exist EC-estimates $\{K_{di}, K_{rj}\}$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$, such that for each $a \in A$ and for each sufficiently small ϱ there exists $a_\varrho \in A_0$ such that the relations:

$$\|a_i - a_{\varrho i}\| \leqslant \varrho_{di} \cdot \|a_i\| \cdot K_{di}; \quad \|\varphi_j(a_\varrho) - x_{\varrho j}\| \leqslant \varrho_{rj} \cdot \|\varphi_j(a_\varrho)\| \cdot K_{rj}$$

hold (x_ϱ denotes as before the result computed by means of V_ϱ or $\{V_\varrho^{(p)}\}$).

11.2. Almost well-behaving algorithms (AWB). We shall now introduce a more general concept. An algorithm V (or $\{V^{(p)}\}$) is said to be *almost well-behaved* in $\{\varphi, A\}$, $A \subset A_0$, with respect to a structure $\Sigma(R_d, R_r)$ if for each $j = 1, 2, \ldots, m$ there exist EC-estimates $\{K_{di}^{(j)}, K_{rj}\}$ such that for each $a \in A$ and for sufficiently small ϱ there exist $a_\varrho^{(j)} \in A_{0j}$ such that the relations:

$$\|a_i - a_{\varrho i}^{(j)}\| \leqslant \varrho_{di} \cdot \|a_i\| \cdot K_{di}^{(j)};$$
$$\|\varphi_j(a_\varrho^{(j)}) - x_{\varrho j}\| \leqslant \varrho_{rj} \cdot \|\varphi_j(a_\varrho^{(j)})\| \cdot K_{rj}$$

hold.

11.3. The optimal error level and NS-algorithms. The $\varkappa$-class of mappings with respect to a structure $\Sigma(R_d, R_r)$ can be defined as follows:

$\varphi: A_0 \subset R_d \to R_r$ is a $\varkappa$-mapping if for each $a \in A_0$ there exist positive numbers

$$\{d_{0i}(a), \alpha_{ij}(a), H_{ij}(a)\}_{\substack{i=1,2,\ldots,l\\ j=1,2,\ldots,m}}$$

such that for each $\hat{a} \in A_0$ the implication:

$$\forall_i\, d_i = \|a_i - \hat{a}_i\| \leqslant d_{0i}(a) \Rightarrow \forall_j\, \|\varphi_j(a) - \varphi_j(\hat{a})\| \leqslant \sum_{i=1}^{l} H_{ij}(a) \cdot d^{\alpha_{ij}(a)}$$

holds.

The *optimal error level* should be now defined separately for each component φ_j of the mapping φ:

$$P(\varphi_j, a, \varrho) = \sum_{i=1}^{l} H_{ij}(a)(\varrho_{di} \cdot \|a_i\|)^{\alpha_{ij}(a)} + \varrho_{rj} \cdot \|\varphi_j(a)\|.$$

An algorithm is said to be *numerically stable* in $\{\varphi, A\}$, with respect to a structure $\Sigma(R_d, R_r)$ if there exist numbers K_j such that for $a \in A$ and for each sufficiently small ϱ the relations:

$$\|\varphi_j(a) - x_{\varrho j}\| \leqslant K_j \cdot P(\varphi_j, a, \varrho), \quad j = 1, 2, \ldots, m,$$

hold.

11.4. Relative error bound and the condition of the problem. If $x_j = \varphi_j(a) \neq 0$ then the bound for the relative error of the solution $x_{\varrho j}$ computed by means of an NS-algorithm can be written in the form

$$\frac{\|x_j - x_{\varrho j}\|}{\|x_j\|} \leqslant K_j \cdot \Big(\sum_{i=1}^{l} C_{ij}(a) \cdot (\varrho_{di})^{\alpha_{ij}(a)} + \varrho_{rj}\Big)$$

where

$$C_{ij}(a) = H_{ij}(a) \frac{\|a_i\|^{\alpha_{ij}(a)}}{\|x_j\|}$$

is the *condition number* for the problem $\{\varphi_j, A\}$ with respect to the ith component a_i of the input data. ($x_j = \varphi_j(a)$, $x_{\varrho j} = V_{\varrho j}[a]$.)

Comments

1. The structure of the spaces R_d, R_r is in practical problems determined to some extent by the meaning of the information contained in the input data and the result. There remains, however, some freedom in applying a more or less refined splitting into components and in the choice of one or another norm in each component. It is possible to define the concept of *equivalent structures* of a given pair R_d, R_r (and of *equivalent substructures*) in such a way that the properties of WB or NS for algorithms, $\varkappa$- or λ-class for mappings remain invariant under the replacement of a given structure by an equivalent one.

2. Several further concepts (as e.g. the RS-concept) or theorems can be extended to the case of a general numerical problem. For example:

An AWB-algorithm in $\{\varphi, A\}$ with respect to a structure $\Sigma(R_d, R_r)$ is NS in $\{\varphi, A\}$ for each equivalent substructure. For the same structure the EC-estimates are $K_j = \max\big(\max_i(K_{di}), K_{rj}\big)$.

3. Let us consider matrices $A\,(k \times l)$, $B\,(l \times m)$, $C = A \cdot B$. If we introduce the structure Σ_1: $R_d = R_{d1} \times R_{d2}$, R_r, $A \in R_{d1}$, $B \in R_{d2}$, $C \in R_r$ with the euclidean norm in R_{di}, R_r, then the "natural" algorithm of matrix multiplication is only NS in $\{\varphi, R_d\}$ with respect to this structure. (It is neither WB nor AWB for Σ_1.) Considering the structure Σ_2: $R_d = R_{d1} \times R_{d2} \times \ldots \times R_{d,m+1}$, $R_r = R_{r1} \times \ldots \times R_{rm}$, $A \in R_{d1}$, $\vec{b}_j \in R_{d,j+1}$, $\vec{c}_j \in R_{rj}$, $\vec{b}_j, \vec{c}_j$ being the columns of the matrices B, C respectively, and introducing the euclidean norm in each component space, we find out that "natural" algorithm for multiplication is AWB in $\{\varphi, R_d\}$ with respect to the structure Σ_2.

References

[1] I. Babuška, *Numerical stability in problems of linear algebra*, SIAM J. NA (1970), pp. 53–77.

[2] F. L. Bauer, *Genauigkeitsfragen bei der Lösung linearer Gleichungssysteme*, ZAMM 46 (1966), pp. 109–121.

[3] D. K. Faddeev, V. N. Faddeeva, *Natural norms in algebraic processes*, Inf. Proc. 68, Amsterdam 1969, pp. 33–39.

[4] W. Kahan, *A survey of error analysis*, IFIP 1971, I, pp. 200–206.
[5] J. Stoer, *Einführung in die numerische Mathematik I*, Springer Verlag, 1972.
[6] V. V. Voevodin, *Rounding errors and stability* (in Russian), MGU, Moscow 1969.
[7] J. H. Wilkinson, *Rounding errors in algebraic processes*, London 1963.

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MULTIVARIATE SECANT METHOD

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We consider the problem of solving a system of nonlinear equations

(1) $$f(x) = 0$$

for $f\colon D \to C^n$, where C^n denotes the n-dimensional complex space and D is an open and convex set in C^n. We assume that f satisfies the following two conditions,

(2)
(i) there exists a simple zero $\alpha = \alpha(f) \in D$;
(ii) $f'(x)$ is a Lipschitz function in D.

We solve (1) by the multivariate secant method—shortly the MS-method—defined as follows. Let $x_i, \ldots, x_{i-n} \in D$ be approximations of α. If the matrices

$$X_i = [\delta x_{i-n}, \ldots, \delta x_{i-1}], \quad F_i = [\delta f_{i-n}, \ldots, \delta f_{i-1}],$$

where $\delta x_j = x_{j+1} - x_j$, $\delta f_j = f_{j+1} - f_j$, $f_j = f(x_j)$, are nonsingular then the next approximation of α in the MS-method is given by the formula:

(3) $$z_i = \varphi(x_i; f) = x_i - X_i \cdot F_i^{-1} \cdot f_i.$$

We can put $x_{i+1} = z_i$ or define x_{i+1} otherwise.

The problems of our interest are,

(i) the convergence and the character of convergence of the MS-method,
(ii) the numerical stability of a chosen algorithm of the MS-method.

For the first problem we got the following result. Let us define

$$d_i = \left|\det\left[\frac{\delta x_{i-n}}{\|\delta x_{i-n}\|}, \ldots, \frac{\delta x_{i-1}}{\|\delta x_{i-1}\|}\right]\right| \quad (d_i \leqslant 1),$$

(4) $$\mathfrak{M}(c, \xi) = \{(x_n, x_{n-1}, \ldots, x_0)\colon x_j \in C^n, d_n \geqslant c\|x_n - x_0\|^{\xi}\},$$

where $\xi \in [0, 1)$, $c \in (0, 1]$.