нениям; погрешность зависит лишь от того, насколько точно выполняются условия согласования на границе раздела сред. К недостаткам можно отнести то, что, во-первых, нужно уметь вычислять фундаментальное решение или функцию Грина, входящие в ядра интегральных операторов, во-вторых, — наличие на Γ угловых точек несколько усложняет решение задачи.

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GLOBALLY CONVERGENT DIFFERENTIAL PROCEDURES FOR SOLVING NONLINEAR SYSTEMS OF EQUATIONS

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1. Introduction

One of the more intriguing ideas put forth with respect to the problem of solving nonlinear equations is that of constructing systems of differential equations with a parameter t such that solutions of the differential equations will converge to solutions of the nonlinear equations for large t. Although such a procedure is a useful one in view of the relative ease with which systems of differential equations can be solved by high speed computers, there are three basic difficulties associated with it. The first is that of establishing the initial data for which solutions of the differential equations converge for large t; the ideal situation being that of global convergence (i.e. for all finite initial data). The second is that of determining whether the limit points of the solutions of the differential equations consist only of solutions of the original nonlinear system, or whether there are additional spurious limit points. And the third is that of determining the rate of convergence. Since the analysis given in this note takes a somewhat different approach to these problems than has been reported in the literature, it does not seem appropriate to cite specific references to the extensive and excellent body of work on this subject. The readers familiar with these problems will have little difficulty in perceiving parallels and implications with respect to the various methods in current use.

We first give a general procedure for constructing families of globally convergent systems of differential equations with a single point of convergence $\boldsymbol{y} = \boldsymbol{0}$. This is achieved by means of Liaponov's method of determining asymptotic stability. Application of this procedure to the problem of solving the system $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{0}$ is shown to yield a globally convergent method if $\Delta(\boldsymbol{x}) = \det(\partial y_i/\partial x_j) \neq 0$ for all $\boldsymbol{x}$. If $\Delta(\boldsymbol{x}) = 0$ on some nonempty set of points, then the same procedure as used in the case $\Delta(\boldsymbol{x}) \neq 0$ implies that the norm of $d\boldsymbol{x}(t)/dt$ can grow without bound. We construct two different classes of procedures that give global convergence for problems with $\Delta(\boldsymbol{x}) = 0$. These procedures converge to all solutions of $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{0}$, but have additional spurious points of convergence that are contained in the set $\Delta(\boldsymbol{x}) = 0$. The set for which $\Delta(x) = 0$ is thus of considerable importance in such procedures. Lastly, we show that the procedures can be adjusted so as to obtain exponential convergence of the norm of $y(x)$ for large t, and, more importantly, for finite t. Specifically, we give a method for which

$$\|y(x(t))\| \leqslant 2\exp(-\tan(kt+c))$$

where k is a given positive constant and c is a constant that is determined by the initial value of $\|y(x(t))\|$.

2. A convergence theorem

Let $x \equiv (x_1, \ldots, x_N)$ be a generic element of an N-dimensional vector space E_N with inner product (x, y) and norm $\|x\| = (x, x)^{1/2}$. The basis for the methods given in this note is the following convergence theorem (global asymptotic stability theorem).

THEOREM. *Every solution of the system of autonomous differential equations*

(2.1) $$\frac{d}{dt}y(t) = -F(y)$$

with finite initial data converges to $y = 0$ *for sufficiently large t provided*

(2.2) $$F(y) = \nabla_W \varphi(W(y)) + U(W(y))$$

for any C^1 *vector-valued function* $U(W)$ *such that*

(2.3) $$U(0) = 0, \quad (W, U(W)) = 0$$

and any scalar-valued function $\varphi(W)$ *such that*

(2.4) $$\varphi(W) = \int_0^1 \{P(\lambda W) - K(\lambda W)\}\frac{d\lambda}{\lambda},$$

where

(2.5) $$W(y) = \nabla_y V(y),$$

$V(y)$ *is any* C^3 *function and* $P(W)$ *and* $K(W)$ *are any* C^2 *functions which satisfy the following conditions:*

(2.6) $$V(y) \geqslant 0,$$

(2.7) $$V(y) = 0 \Leftrightarrow y = 0,$$

(2.8) $$\nabla_y V(y) = 0 \Leftrightarrow y = 0,$$

(2.9) $$K(W) \leqslant 0,$$

(2.10) $$K(W) = 0 \Leftrightarrow W = 0,$$

(2.11) $$P(W) \geqslant 0,$$

(2.12) $$P(0) = 0.$$

Proof. We first note that (2.9)–(2.12) imply

(2.13) $$\nabla_W(P-K)|_{W=0} = 0,$$

while (2.6)–(2.8) imply $V(y)$ tends to infinity as $\|y\|$ tends to infinity. Since $V(y) \in C^3$, we have $W(y) = \nabla_y V(y) \in C^2$, and hence (2.2), (2.4) and the continuity conditions satisfied by $U(W)$, $P(W)$ and $K(W)$ show that $F(y) \in C^1$. It also follows from (2.2)–(2.5), (2.8) and (2.13) that

$$F(y) = 0 \Leftrightarrow W(y) = 0 \Leftrightarrow y = 0,$$

and hence $y = 0$ is the only critical point of the system (2.1). A straightforward calculation and use of (2.1) gives us

(2.14) $$\frac{d}{dt}V(y(t)) = (\nabla_y V(y), dy(t)/dt) = -(\nabla_y V, F),$$

and hence (2.6)–(2.8) show that $V(y)$ will be a global Liaponov function for the system (2.1) that implies the global asymptotic stability of $y = 0$ provided [1]

(2.15) $$\psi(y) = (\nabla_y V(y), F(y))$$

satisfies the conditions

(2.16) $$\psi(y) \geqslant 0,$$

(2.17) $$\psi(y) = 0 \Leftrightarrow y = 0.$$

Now, (2.5) and (2.8) imply that $W(y) = 0 \Leftrightarrow y = 0$, while (2.2) shows that we actually have $F(y) = F^*(W(y))$, where $F^*(W)$ is of class C^1 in W. Accordingly (2.15)–(2.17) can be rewritten in the equivalent forms

(2.18) $$\psi(y) = \psi^*(W(y)) = (W(y), F^*(W(y))),$$

(2.19) $$\psi^*(W) \geqslant 0,$$

(2.20) $$\psi^*(W) = 0 \Leftrightarrow W = 0.$$

We have shown in a previous paper [2] that the general solution of the inequality

(2.21) $$(W, F^*(W)) + K(W) \geqslant 0,$$

for $F^*(W) \in C^1$ and $K(W) \in C^2$, is given by

(2.22) $$F^*(W) = \nabla_W \varphi(W) + U(W),$$

where (2.3), (2.4), (2.11), $P(0) = K(0) = 0$ and the given continuity conditions on $U(W)$, $P(W)$ and $K(W)$ are satisfied. In fact, the relation between F^*, K and P is

(2.23) $$(W, F^*(W)) + K(W) = P(W) \geqslant 0,$$

and hence equality holds in (2.21) if and only if $P(W) \equiv 0$. It thus follows, on combining (2.18) with (2.21) and (2.23), that

(2.24) $$\psi^*(W) \geqslant -K(W),$$

with equality holding if and only if $P(W) \equiv 0$. The inequalities (2.19) and (2.20) are thus satisfied provided $K(W)$ is such that (2.9) and (2.10) hold. We must also have satisfaction of (2.12) since $P(\mathbf{0}) = K(\mathbf{0}) = 0$. The function $V(y)$ is thus established as a Liaponov function for the system (2.1) with $dV(y(t))/dt = 0$ if and only if $y = \mathbf{0}$, and the theorem is established.

It is of interest to note that the above theorem characterizes the most general system of autonomous differential equations (2.1) for which $F(y)$ is of class C^1 and $y = \mathbf{0}$ is a globally asymptotically stable critical point with a global Liaponov function $V(y)$ of class C^3.

3. Differential methods for solving $y(x) = 0$

As before, let x be a generic element of E_N and let $y(x)$ be a C^1 mapping of E_N into E_N. Further, let $x(t)$ be a C^1 mapping of $[0, \infty)$ into E_N, then $y^*(t) = y(x(t))$ is a C^1 mapping of $[0, \infty)$ into E_N. With $F(y^*(t))$ chosen in accordance with the hypotheses of the convergence theorem, the system of autonomous differential equations

$$\frac{d}{dt} y^*(t) = -F(y^*(t)) \tag{3.1}$$

has a solution $y^*(t)$, by forward stepping procedures for all finite initial data

$$y_0^* = y^*(0), \tag{3.2}$$

which converges to the zero vector for sufficiently large t. What we want, however, is a system of equations for the determination of $x(t)$, rather than $y^*(t) = y(x(t))$, which converges to solutions of

$$y(x) = \mathbf{0} \tag{3.3}$$

for sufficiently large t. A formal substitution of $y^*(t) = y(x(t))$ into (3.1) yields

$$\sum_{j=1}^{N} \frac{\partial y_i}{\partial x_j} \frac{d}{dt} x_j(t) = -F_i(y(x(t))) \equiv -F_i^*(x(t)), \tag{3.4}$$

from which it is clear that a system of differential equations can be obtained for $x(t)$. The convergence properties of the solutions of (3.4) are, in general, however, quite different from those of solutions of (3.1). In fact, there are two distinct cases which arise.

C a s e 1. *Local invertibility of* $y(x) = \mathbf{0}$. This case is delineated by satisfaction of the condition

$$\det(\partial y_i/\partial x_j) \neq 0 \tag{3.5}$$

for every x in E_N, and that $y(x)$ is of class C^2. Since satisfaction of (3.5) is sufficient for local invertibility of $y(x) = g$, we refer to this case as locally invertible. Under satisfaction of (3.5), the system (3.4) can be written in the equivalent form

$$\frac{d}{dt} x_i(t) = -\sum_{j=1}^{N} Y_{ij}(x) F_j^*(x), \tag{3.6}$$

where $Y_{ij}(x)$ are the entries of the inverse of the matrix $((\partial y_i/\partial x_j))$. Since $y(x) \in C^2$, the functions $Y_{ij}(x) \in C^1$, and the right-hand sides of (3.6) are C^1 functions of x.

Let $V(y)$ be the Liaponov function for the system (3.1) which is used in the construction of the vector-valued function $F(y)$. We then have

$$V^*(x) = V(y(x)) \geqslant 0, \tag{3.7}$$

$$V^*(x) = 0 \Leftrightarrow y(x) = \mathbf{0}, \tag{3.8}$$

$$\frac{\partial V^*}{\partial x_i} = \sum_j \frac{\partial V}{\partial y_j} \frac{\partial y_j}{\partial x_i} = 0 \Leftrightarrow \frac{\partial V}{\partial y_j} = 0 \Leftrightarrow y(x) = 0, \tag{3.9}$$

from (2.6)–(2.8), and

$$\frac{dV^*}{dt} = \left(\nabla_x V^*, \frac{d}{dt} x\right) = \left(\nabla_y V, \frac{d}{dt} y\right) \leqslant K(y(x)) = K^*(x) \leqslant 0, \tag{3.10}$$

$$K^*(x) = 0 \Leftrightarrow y(x) = \mathbf{0}, \tag{3.11}$$

from (2.10), (2.14), (2.15) and (2.24). Thus, $V^*(x)$ is a global Liaponov function for the system (3.6) whose only critical points are those x for which $y(x) = \mathbf{0}$. Convergence of all solutions of (3.6) to solutions of $y(x) = \mathbf{0}$ for sufficiently large t thus follows for all finite initial data $x_0 = x(0)$. In fact, it follows that the solution of $y(x) = \mathbf{0}$ is unique in E_N.

C a s e 2. *Noninvertibility*. This case is characterized by

$$\det(\partial y_i/\partial x_j) = 0$$

at one or more points of E_N. Analysis of this case is most easily pursued by introducing the following point sets:

$$\mathscr{D} = \{x \in E_N |\ \det(\partial y_i/\partial x_j) = 0\}, \tag{3.12}$$

$$\mathscr{J} = \{x \in E_N |\ \|\nabla_x V^*\| = 0,\ \|y(x)\| \neq 0\}, \tag{3.13}$$

$$\mathscr{X} = \{x \in E_N |\ y(x) = \mathbf{0}\}, \tag{3.14}$$

where $V^*(x) = V(y(x))$ and $V(y)$ is the Liaponov function for the system (3.1) that is used in the construction of $F(y)$. Since

$$\frac{\partial V^*}{\partial x_i} = \sum_{j=1}^{N} \frac{\partial V}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

and $\nabla_y V = \mathbf{0} \Leftrightarrow y(x) = \mathbf{0} \Leftrightarrow x \in \mathscr{X}$, we see that $x \in \mathscr{J}$ only if $x \in \mathscr{D}$; that is

$$\mathscr{J} \subset \mathscr{D}. \tag{3.15}$$

It also follows that

$$\frac{d}{dt} V^* = \left(\nabla_x V^*, \frac{d}{dt} x\right) = \left(\nabla_y V, \frac{d}{dt} y\right) \leqslant K(y(x)) = K^*(x) \leqslant 0, \tag{3.16}$$

$$K(y(x)) = K^*(x) = 0 \Leftrightarrow x \in \mathscr{X}. \tag{3.17}$$

However, for each point $x \in \mathscr{J}$, $\|\nabla_x V^*\| = 0$. Since $\mathscr{X} \cap \mathscr{J}$ is empty, it follows from (3.16) that $\left\|\frac{d}{dt}x\right\|$ is unbounded for all $x \in \mathscr{J}$. An integration of the system (3.4) can thus lead to unbounded values for dx_i/dt and global asymptotic stability of the system (3.4) can fail to obtain. Notice that we have only proved that it is possible for global asymptotic stability of (3.1) to fail to provide global asymptotic stability of (3.4); not that convergence of (3.4) necessarily fails. In this regard, it is useful to note that the inclusion relation (3.15) shows that it is the point set $\mathscr{D}$ in which troublesome points can occur, if indeed there are any for a given set of initial data. This is particularly convenient since the point set $\mathscr{D}$ is characterized by satisfaction of only the one relation $\det(\partial y_i/\partial x_j) = 0$, as opposed to the relations $\nabla_x V^* = \mathbf{0}$, $y(x) \neq \mathbf{0}$ which characterize $\mathscr{J}$.

The possibility of unbounded derivatives occurring in the course of solving the system (3.4) will now be circumvented, at the cost, however, of introducing spurious points of convergence. Consider the autonomous system

$$\frac{d}{dt}x(t) = -\nabla_x \psi\big(V(y(x))\big) = -\frac{d\psi}{dV^*}\nabla_x V^*, \tag{3.18}$$

where $V(y)$ is any C^2 function that satisfies the conditions (2.6)–(2.8) and $\psi(V)$ is a C^2 function of V such that

$$d\psi/dV \geqslant 0 \tag{3.19}$$

for all non-negative V and

$$d\psi/dv = 0 \Leftrightarrow V = 0. \tag{3.20}$$

It then follows that

$$\frac{d}{dt}V^*(x) = \left(\nabla_x V^*, \frac{d}{dt}x\right) = -\frac{d\psi}{dV^*}(\nabla_x V^*, \nabla_x V^*) \tag{3.21}$$

$$= -\frac{d\psi}{dV^*}\|\nabla_x V^*\|^2 \leqslant 0,$$

and hence the functions dV^*/dt and dx/dt vanish if and only if $x \in \mathscr{X} \cup \mathscr{J}$. In particular, $\|dx/dt\|$ is finite for any solution of (3.18). Accordingly, every solution of (3.18) will converge to a point of relative minimum of $V^*(x)$ or remain at points in $\mathscr{X} \cup \mathscr{J}$ if it happens to start at points in $\mathscr{X} \cup \mathscr{J}$. Since $\mathscr{X}$ consists only of points of absolute minimum of V^* (i.e., $V^*(x) = 0 \Leftrightarrow x \in \mathscr{X}$), while $\mathscr{J}$ contains both points of relative maximum and relative minimum of $V^*(x)$, any solution of (3.18) that does not start in $\mathscr{X} \cup \mathscr{J}$ will converge to a point in $\mathscr{X} \cup \mathscr{M}$, where $\mathscr{M}$ is the subset of $\mathscr{J}$ which gives relative minima of $V^*(x)$. We thus have a convergent method for all initial data belonging to the complement of $\mathscr{J}$, but at the expense of convergence to the spurious points $x \in \mathscr{M}$. In particular, we have a convergent method for any initial data belonging to the complement of $\mathscr{D}$ since $\mathscr{J} \subset \mathscr{D}$. This latter characterization is simpler since it only requires calculation of the zeros of the single function $\det(\partial y_i/\partial x_j)$. On the other hand, characterization of the set $\mathscr{J}$ requires that we find all zeros of the vector system of equations $\nabla_x V^*(x) = \mathbf{0}$, and this is of the same order of difficulty as solving the original system $y(x) = \mathbf{0}$.

4. Generalizations

It was shown in the last section that there are many methods for solving $y(x) = \mathbf{0}$ when $y(x) = g$ is locally invertible for every $x \in E_N$. On the other hand, we gave only one method in the event that $y(x) = g$ was not invertible for some $x \in E_N$. Two general classes of methods are obtained in this section for the noninvertible case.

Let $V(y)$ be a C^2 scalar-valued function such that

$$V(y) \geqslant 0, \quad V(y) = 0 \Leftrightarrow y = \mathbf{0}, \tag{4.1}$$

$$\nabla_y V(y) = 0 \Leftrightarrow y = \mathbf{0}, \tag{4.2}$$

and define the function $V^*(x)$ by

$$V^*(x) = V\big(y(x)\big) \tag{4.3}$$

for $y(x)$ a given C^2 vector-valued function. Let

$$F^*(x) = F\big(W(x)\big) \tag{4.4}$$

be defined by (2.2), where $W(x)$ is now given by

$$W(x) = \nabla_x V^*(x) \tag{4.5}$$

and (2.3), (2.4), (2.9)–(2.13) hold. We consider the system of autonomous differential equations

$$\frac{d}{dt}x(t) = -F^*(x), \tag{4.6}$$

and use of this system shows that

$$\frac{d}{dt}V^*(x) = -(\nabla_x V^*, F^*(x)) = -\big(W, F(W)\big). \tag{4.7}$$

The same reasoning as that given in Section 2 now yields

$$\frac{d}{dt}V^*(x) \leqslant K(W) \leqslant 0 \tag{4.8}$$

with

$$K(W) = 0 \Leftrightarrow W = \mathbf{0}. \tag{4.9}$$

It also follows from (4.4) and the results established in Section 2 that

$$F^*(x) = F\big(W(x)\big) = \mathbf{0} \Leftrightarrow W = \mathbf{0}, \tag{4.10}$$

and hence dV^*/dt vanishes only at the critical points of the system (4.6) which are the points $W(x) = \mathbf{0}$. However, we saw in the last section that $W(x) = \nabla_x V^*(x) = \mathbf{0}$ holds for all $x \in \mathscr{X} \cup \mathscr{J}$. Accordingly, the same reasoning as used in Section 3 shows that every solution of the system (4.6) with initial data in the complement of $\mathscr{J}$ will converge to a point in $\mathscr{X} \cup \mathscr{M}$. It also follows from the fact that the set $\mathscr{X}$ is the set of absolute minima of $V^*(x)$, that every isolated point $\bar{x}$ of $\mathscr{X}$ belongs to an open neighborhood $\mathscr{N}(\bar{x})$ of initial data which contains $\bar{x}$ and is such that every solution of (4.6) with initial data in $\mathscr{N}(\bar{x})$ will converge to $\bar{x}$. It is also clear that if $\mathscr{X}$ contains an arcwise connected subset $\mathscr{A}$, then there exists an arcwise connected open set $\mathscr{N}(\mathscr{A})$ which contains $\mathscr{A}$ as a proper subset such that every solution of (4.6) with initial data in $\mathscr{N}(\mathscr{A})$ will converge to a point on the boundary of $\mathscr{A}$ if the initial point does not belong to $\mathscr{A}$, and will remain fixed in $\mathscr{A}$ if the initial data point belongs to $\mathscr{A}$.

Although these results provide significant generalizations of those obtained in Section 3, they still implicitly contain the difficulty that one must characterize the set of points $\mathscr{J}$, and this in turn requires that we solve the vector system of equations $\nabla_x V^*(x) = \mathbf{0}$. We therefore present an alternative collection of methods which does not involve this difficulty.

Let $y(x)$ be a given C^1 vector-valued function and let

$$J(x) = \big((\partial y_i(x)/\partial x_j)\big) \tag{4.11}$$

denote the Jacobian matrix of $y(x)$ with respect to x. For the remainder of this section we view x as a column matrix with N entries and x^{T}, the transpose of x, as a row matrix. Further, let $\mathbf{C}(x)$ denote the matrix of cofactors of the matrix $J(x)$, so that

$$J(x)\mathbf{C}(x)^{\mathrm{T}} = \Delta(x)\mathbf{E}, \tag{4.12}$$

where $\mathbf{E}$ denotes the N-by-N identity matrix and

$$\Delta(x) = \det\big(J(x)\big) = \det(\partial y_i/\partial x_j). \tag{4.13}$$

Choose $F(y)$ in accordance with the convergence theorem given in Section 2 in terms of the functions $U(W)$, $V(y)$, $P(W)$, $K(W)$, where we now have $W = \nabla_y V(y)$. A convenient system of autonomous differential equations to be used under these conditions is

$$\frac{d}{dt}x(t) = -\Delta(x)\mathbf{C}(x)^{\mathrm{T}}F^*(x) \tag{4.14}$$

where $F^*(x) = F\big(y(x)\big)$. We then have

$$\frac{d}{dt}y = J(x)\frac{d}{dt}x = -\Delta(x)J(x)\mathbf{C}(x)^{\mathrm{T}}F^*(x) = -\big(\Delta(x)\big)^2F(y), \tag{4.15}$$

when (4.12) is used. The only difference between (4.15) and (2.1) is the factor $\big(\Delta(x)\big)^2$ on the right-hand side of (4.15). Hence the critical points of the system (4.15) consist of those for which $y = y(x) = \mathbf{0}$ and, in addition, those for which $\Delta(x) = 0$; that is, the set of points $\mathscr{X} \cup \mathscr{D}$. Likewise, since the matrix $\mathbf{C}(x)$ is singular only if $\Delta(x) = 0$, the system (4.14) possesses exactly the same critical points, namely $\mathscr{X} \cup \mathscr{D}$. A straightforward calculation and the results established in Section 2 give

$$\frac{d}{dt}V(y) = (\nabla_y V)^{\mathrm{T}}\frac{d}{dt}y = -\big(\Delta(x)\big)^2(\nabla_y V, F) \leqslant \big(\Delta(x)\big)^2K(\nabla_y V) \leqslant 0, \tag{4.16}$$

$$\frac{d}{dt}V(y) = 0 \Leftrightarrow y = \mathbf{0} \quad \text{or} \quad \Delta(x) = 0, \tag{4.17}$$

or

$$\frac{d}{dt}V^*(x) = (\nabla_y V)^{\mathrm{T}}J\frac{d}{dt}x = -\big(\Delta(x)\big)^2(\nabla_y V)^{\mathrm{T}}F \leqslant \big(\Delta(x)\big)^2K(\nabla_y V) \leqslant 0. \tag{4.18}$$

It thus follows that every solution of the system (4.14) will converge to a point in the set $\mathscr{X} \cup \mathscr{D}$ for sufficiently large t. In addition, since $V^*(x)$ achieves its absolute minimum of zero on the set $\mathscr{X}$ (i.e., $V^*(x) = V\big(y(x)\big)$), each isolated point $\bar{x}$ of $\mathscr{X}$ which does not belong to $\mathscr{D}$ is contained as an interior point of a neighborhood $\mathscr{N}(\bar{x})$ that contains no points of $\mathscr{D}$, and for which every solution of (4.14) which starts in $\mathscr{N}(\bar{x})$ will converge to $\bar{x}$.

Although this method does not require that we characterize the set $\mathscr{J}$, it has the disadvantage that every point of $\mathscr{D}$ is now a spurious attractive point. This can be particularly troublesome since the set $\mathscr{D}$ can divide E_N into two or more parts, in which case zeros of $y(x) = \mathbf{0}$ which lie in one part can not be approached by points starting with initial data in another part. It would thus appear that the previous collection of methods generated by (4.6) may often be preferable since $\mathscr{J} \subset \mathscr{D}$. Further, $\mathscr{J}$ is usually significantly smaller than $\mathscr{D}$, and $\mathscr{J}$ is often discrete while $\mathscr{D}$ is not, in general. This latter observation is particularly important in those cases where $\mathscr{D}$ divides E_N into two or more parts, for $\mathscr{J}$ may not have this property.

5. Examples and convergence rates

We construct specific examples in this section for which the rate of convergence of the method can be determined. Although we confine our analysis to the system studied in Section 2, it is evident that these same considerations can be applied to the convergence problem for either the system (4.6) or (4.14).

We take

$$V(y) = \tfrac{1}{2}(y, y) = \tfrac{1}{2}\|y\|^2, \tag{5.1}$$

so that (2.5) gives

$$W(y) = y. \tag{5.2}$$

Accordingly, if we select $F(y)$ in accordance with the convergence theorem given in Section 2, and take $U(y) = \mathbf{0}$, then (2.2), (2.5) and (5.2) give

$$F(y) = \nabla_y \varphi(y) \tag{5.3}$$

and we have

$$\frac{d}{dt}y = -\nabla_y\varphi(y) = -\nabla_y\int_0^1\{P(\lambda y) - K(\lambda y)\}\frac{d\lambda}{\lambda}, \tag{5.4}$$

$$\frac{d}{dt}V = K(y) - P(y) \leqslant K(y) \leqslant 0 \tag{5.5}$$

since (2.9)–(2.13) hold. Since $K(y)$ is at our disposal, subject only to the conditions $K(y) \leqslant 0$, $K(y) = 0 \Leftrightarrow y = \mathbf{0}$, we are free to select $K(y)$ by

(5.6) $$K(y) = -Q(\tfrac{1}{2}\|y\|^2) = -Q(V),$$

where $Q(V)$ is a C^2 function of its argument such that

(5.7) $$Q(V) \geqslant 0 \text{ for } V \geqslant 0, \quad Q(0) = 0 \Leftrightarrow V = 0.$$

In this event, (5.5) becomes

(5.8) $$\frac{d}{dt}V = -Q(V) - P(y) \leqslant -Q(V)$$

since $P(y) \geqslant 0$. It thus follows that every solution of (5.4) will have the property

(5.9) $$\|y(t)\|^2 \leqslant 2v(t),$$

where $v(t)$ satisfies the differential equation

(5.10) $$\frac{dv}{dt} = -Q(v)$$

subject to the initial condition

(5.11) $$v(0) = \tfrac{1}{2}\|y(0)\|^2.$$

We note in passing that equality obtains in (5.9) if and only if we choose the function $P(y)$ such that $P(y) \equiv 0$.

It is now a straightforward problem to choose the function $Q(V)$ in such a way to obtain various rates of convergence. An obvious first choice is

(5.12) $$Q(V) = kV, \quad k > 0,$$

which obviously satisfies the conditions (5.7). In this case, (5.10) becomes $dv/dt = -kv$, and hence (5.9) and (5.11) yield

(5.13) $$\|y(t)\|^2 \leqslant 2\|y(0)\|^2 \exp(-kt);$$

that is, we have exponential convergence. In particular, if we take $P(y) \equiv 0$, then $F(y) = -ky$ and we obtain strict equality in (5.13). Application of these results to the problem of solving $y(x) = \mathbf{0}$ by means of the system (4.14) and (4.15) gives

(5.14) $$\frac{d}{dt}x(t) = -\Delta(x)\mathbf{C}(x)^{\mathrm{T}}y(x),$$

(5.15) $$\frac{d}{dt}y(x(t)) = -(\Delta(x))^2 y(x).$$

The system (5.15) is seen to be quite similar to the Davidenko–Branin [3] method $(dy(x)/dt = \operatorname{sign}(\Delta(x))y)$ with the exception that (5.15) has a C^1 right-hand side, while the function $\operatorname{sign}(\Delta(x))$ which occurs in the Davidenko system has a jump discontinuity on the set $\mathscr{D}$, and all of the attendant troubles which such discontinuities cause.

A more striking example is that provided by the choice

(5.16) $$Q(V) = kV(1 + (\ln V)^2), \quad k > 0$$

which clearly satisfies the conditions (5.7). In this case, (5.10) gives

(5.17) $$\frac{dv}{dt} = -kv(1 + (\ln v)^2),$$

and hence (5.11) and (5.17) yield

(5.18) $$v(t) = \exp(-\tan(kt + c)),$$

(5.19) $$-\pi/2 \leqslant c = -\tan^{-1}\big(\ln(\tfrac{1}{2}\|y(0)\|^2)\big) \leqslant \pi/2.$$

Thus, (5.9) shows that

(5.20) $$\|y(t)\|^2 \leqslant 2\exp(-\tan(kt + c));$$

that is, we obtain *exponential convergence in finite time* $T = \left(\frac{\pi}{2} - c\right)k^{-1}$ where c is given by (5.19).

If we wish to apply this result to the problem of solving $y(x) = \mathbf{0}$ by means of the systems (4.14) and (4.15), then we must compute $F(y)$, and hence $\varphi(y)$. Now for $P(y) = 0$,

$$\varphi(y) = -\int_0^1 K(\lambda y)\frac{d\lambda}{\lambda} = +\int_0^1 Q(\lambda^2 V)\frac{d\lambda}{\lambda}$$

$$= kV\int_0^1 \{1 + (\ln \lambda^2 V)^2\}\lambda\, d\lambda = \frac{k}{2}V\{3 - 2\ln V + (\ln V)^2\}$$

and hence (5.3) gives

(5.21) $$F(y) = \frac{k}{2}\{1 + (\ln V)^2\}y.$$

Thus (4.14) and (4.15) yield

(5.22) $$\frac{d}{dt}x(t) = -\frac{k\Delta(x)}{2}\{1 + (\ln V)^2\}\mathbf{C}(x)^{\mathrm{T}}y(x),$$

(5.23) $$\frac{d}{dt}y(x) = -\frac{k}{2}(\Delta(x))^2\{1 + (\ln V)^2\}y(x)$$

and

(5.24) $$\frac{dV}{dt} = \left(y, \frac{dy}{dt}\right) = -k(\Delta(x))^2\{1 + (\ln V)^2\}V.$$

Thus, if the point set $\mathscr{N} \equiv \{x \in E_N | V(y(x)) = \frac{1}{2}\|y\|^2 < R^2\}$ contains no points such that $\Delta(x) = 0$, then there exists a number $\delta > 0$ such that

(5.25) $$\delta = \min_{x \in \mathscr{N}}(\Delta(x))$$

and (5.25) yields

(5.26) $$\frac{dV}{dt} \leqslant -\delta kV\{1 + (\ln V)^2\} \leqslant 0.$$

Hence, every solution of (5.22) which starts in $\mathcal{N}$ will converge such that

$$\|y(x)\|^2 \leqslant 2\exp(-\tan(\delta kt+c)) \tag{5.27}$$

for

$$c = -\tan^{-1}\left(\ln\left(\tfrac{1}{2}\|y(0)\|^2\right)\right). \tag{5.28}$$

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DIFFERENTIAL PROCEDURES FOR SYSTEMS OF IMPLICIT RELATIONS AND IMPLICITLY COUPLED NONLINEAR BOUNDARY VALUE PROBLEMS

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1. Introduction and preliminary considerations

Theory, in the form of the implicit function theorem, is quite specific about solvability of implicit systems of relations. Numerical procedures for the realization of the solutions, when the conditions of the implicit function theorem are met, is quite another matter. The purpose of this note is to give families of differential procedures for solving systems of implicit relations.

Let $\boldsymbol{x}$ denote a vector (column matrix) in n-dimensional number space E_n and let $\boldsymbol{\alpha}$ denote a vector in m-dimensional number space E_m. Let $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\alpha})$ be a given vector valued function that is defined for all $\boldsymbol{x}$ in a given n-dimensional region R_n of E_n and all $\boldsymbol{\alpha}$ in E_m and which takes its values in E_m. Thus, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\alpha})$ is a mapping of $R_n \times E_m$ into E_m. We further assume that $\boldsymbol{f}$ is continuous over its domain of definition and that its matrices of partial derivatives

$$\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{\alpha}) = \nabla_x \boldsymbol{f}, \quad \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{\alpha}) = \nabla_\alpha \boldsymbol{f} \tag{1.1}$$

are also continuous on the domain of definition of $\boldsymbol{f}$. To be more specific, let i, j, k be indices which can take on values from 1 through n and let a, b, c be indices that can take on values from 1 through m. We then have

$$\boldsymbol{A} = ((\partial f_a/\partial x_i)), \quad \boldsymbol{B} = ((\partial f_a/\partial \alpha_b)),$$

so that $\boldsymbol{A}$ is an m-by-n matrix and $\boldsymbol{B}$ is an m-by-m matrix.

The problem we wish to solve is that of constructing differential procedures for obtaining $\boldsymbol{\alpha} = \boldsymbol{\varphi}(\boldsymbol{x})$ as a vector valued function of $\boldsymbol{x}$ such that

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x})) = \boldsymbol{0} \tag{1.2}$$

is satisfied at all points $\boldsymbol{x}$ of E_n for which such solutions exist. The implicit function theorem tells us that if there are elements $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{\alpha}}$ such that

$$\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\alpha}}) = \boldsymbol{0} \tag{1.3}$$

and $\det(\boldsymbol{B}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\alpha}})) \neq 0$, then there exists an n-dimensional neighborhood $N(\bar{\boldsymbol{x}})$ of