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ON A MAXIMUM PRINCIPLE FOR NONLINEAR EQUATIONS

TADEUSZ STYŚ

Institute of Informatics, University of Warsaw, Warsaw, Poland

Introduction

In this paper we study a maximum principle in the sense of R. S. Varga's definition (cf. [1], another formulation of this principle, cf. [2], [3], [4]) for nonlinear systems of algebraic equations and for nonlinear systems of ordinary differential equations. The class of equations which we study arises from an approximation of nonlinear elliptic equations and nonlinear parabolic equations.

In the first part of this paper we show that R. S. Varga's lemma (cf. [1]) on a maximum principle can be also applied to nonlinear systems of algebraic equations. Using R. S. Varga's results, we prove the maximum principle in the sense of Definition 2 for systems of equations of the form (1). Next, we show that the difference scheme (7), (8) satisfies the maximum principle in the sense of Definitions 1 and 2. In the second part we consider the maximum principle in the sense of Definitions 3 and 4 for nonlinear systems of ordinary differential equations. Using this principle, we prove convergence of the method of lines for a nonlinear parabolic equation.

1. A maximum principle for nonlinear systems of algebraic equations

We consider the following system of nonlinear equations:

(1)
$$f(x) = b, \quad b \in R^N,$$

where

$$x = (x_1, x_2, \ldots, x_N) \in R^N, \quad b = (b_1, b_2, \ldots, b_N)^* \in R^N,$$
$$f(x) = (f^1(x), f^2(x), \ldots, f^N(x))^*, \quad f \in C^1(R^N), \quad f(0) = 0.$$

Let $M(x)$ be a matrix with entries $M_{ik}(x) = f^i_{x_k}(xQ_i)$, $0 < Q_i < 1$, $i, k = 1, 2, \ldots, N$ and let S denote the subspace of R^N spanned by the vectors $\delta_j \in \{e_1, e_2, \ldots, e_N\}$, $j = 1, 2, \ldots, r$, $1 \leqslant r \leqslant N$, $e_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{Ni})$, $i = 1, 2, \ldots, N$. The project of a vector x on the subspace S is denoted by $P_S x$, where $P_S =$ diagonal $(d_1, d_2, \ldots$ $\ldots, d_N)$, $d_i = 1$ if $e_i \in S$ and $d_i = 0$ if $e_i \notin S$.

DEFINITION 1. A vector-function $f(x)$ *satisfies a maximum principle with respect to the subspace* S (in symbols, $f\in\mathfrak{M}_S$) if and only if every solution y of $f(x)=P_Sb$ satisfies the inequality

$$\|y\|\leqslant\|P_Sb\|\quad\text{for any } b\in R^N, \tag{2}$$

where $\|x\|=\max|x_i|$.

If for every $b\in R^N$ and for every solution $\mathring{x}$ of $f(x)=P_Sb$ there exists a matrix $A(\mathring{x})$ such that $f(\mathring{x})=A(\mathring{x})\mathring{x}$ and $A(\mathring{x})$ satisfies the maximum principle given in [1], then $f\in\mathfrak{M}_S$. It is clear that we can take $A(\mathring{x})=M(\mathring{x})$ and then $f\in\mathfrak{M}_S$ if $M\in\mathfrak{M}_S$.

In [4] G. T. Mcallister proved a difference analogue of a maximum principle in the form of an *a priori* estimation for a quasilinear elliptic equation. This maximum principle directly implies convergence difference methods. Below, we also consider a maximum principle as an *a priori* estimation of the form (3) for nonlinear systems of equations (1).

DEFINITION 2. A vector-function $f(x)$ *satisfies a maximum principle in the sense of an a priori estimation with respect to the subspace* S *and a number* K (in symbols, $f\in\mathfrak{M}_S(K)$) if and only if every solution y of $f(x)=b$ satisfies the inequality

$$\|y\|\leqslant\|P_Sb\|+K\|(E-P_S)b\|\quad\text{for any } b\in R^N, \tag{3}$$

where E is the unit matrix.

THEOREM 1. *If for every* $b\in R^N$ *and for every solution* $\mathring{x}$ *of* $f(x)=b$ *there exists a monotone matrix* $A(\mathring{x})$ *and a vector* $\hat{\alpha}\in R^N$ *such that*

$$\begin{aligned}&f(\mathring{x})=A(\mathring{x})\mathring{x},\\&A(\mathring{x})\xi\geqslant P_S\xi,\qquad \xi=(1,1,\ldots,1)^*,\\&A(\mathring{x})\hat{\alpha}\geqslant(E-P_S)\xi,\qquad \|\hat{\alpha}\|\leqslant K,\end{aligned} \tag{4}$$

then $f\in\mathfrak{M}_S(K)$.

Proof. Let $U=g-\mathring{x}$, $V=g+\mathring{x}$, where $g=\|P_Sb\|\xi+\|(E-P_S)b\|\hat{\alpha}$. It is easy to verify that

$$\begin{aligned}AU&=Ag-A\mathring{x}=\|P_Sb\|A\xi+\|(E-P_S)b\|A\hat{\alpha}-b\\&\geqslant\|P_Sb\|P_S\xi+\|(E-P_S)b\|(E-P_S)\xi-b\geqslant 0,\\AV&=Ag+A\mathring{x}=\|P_Sb\|A\xi+\|(E-P_S)b\|A\hat{\alpha}+b\\&\geqslant\|P_Sb\|P_S\xi+\|(E-P_S)b\|(E-P_S)\xi+b\geqslant 0.\end{aligned}$$

Since the matrix $A(\mathring{x})$ is monotone, then $g-\mathring{x}\geqslant 0$ and $g+\mathring{x}\geqslant 0$. Therefore we have $\|\mathring{x}\|\leqslant\|P_Sb\|+K\|(E-P_S)b\|$.

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Let us consider the following boundary value problem:

$$G(\alpha,\beta,u,u_\alpha,u_\beta,u_{\alpha\alpha},u_{\beta\beta})=\varphi(\alpha,\beta),\qquad(\alpha,\beta)\in\Omega, \tag{5}$$

$$A(\alpha,\beta)\frac{du}{dn}+B(\alpha,\beta)u=\psi(\alpha,\beta),\qquad(\alpha,\beta)\in\partial\Omega, \tag{6}$$

where $\Omega=\{(\alpha,\beta)\colon 0<\alpha<a_1,\ 0<\beta<a_2\}$.

ASSUMPTIONS.

1°. The function $G(\alpha,\beta,r,s,t,w,z)$ is defined in $\bar{\Omega}\times R^5$ and continuously differentiable with respect to the variables r,s,t,w,z.

2°. $G(\alpha,\beta,0,0,0,0,0)=0$ for $(\alpha,\beta)\in\Omega$ and $G_r(\alpha,\beta,r,s,t,w,z)\leqslant 0$ for $(\alpha,\beta,r,s,t,w,z)\in\Omega\times R^5$.

3°. There exist functions $\mu(r,s,t,w,z)>0$, $\nu(r,s,t,w,z)>0$, $K(r,s,t,w,z)$, $L(r,s,t,w,z)$ and constants $M_1>0$, $M_2>0$ such that

$$\begin{aligned}&G_w(\alpha,\beta,r,s,t,w,z)\geqslant\mu(r,s,t,w,z),\\&G_z(\alpha,\beta,r,s,t,w,z)\geqslant\nu(r,s,t,w,z),\\&|G_s(\alpha,\beta,r,s,t,w,z)|\leqslant K(r,s,t,w,z),\\&|G_t(\alpha,\beta,r,s,t,w,z)|\leqslant L(r,s,t,w,z),\\&K(r,s,t,w,z)\leqslant M_1\mu(r,s,t,w,z),\\&L(r,s,t,w,z)\leqslant M_2\nu(r,s,t,w,z),\end{aligned}$$

for $(\alpha,\beta,r,s,t,w,z)\in\Omega\times R^5$.

4°. The functions $A(\alpha,\beta)\geqslant 0$, $B(\alpha,\beta)\geqslant 0$, $\varphi(\alpha,\beta)$, $\psi(\alpha,\beta)$ are bounded in $\partial\Omega$ and Ω, respectively. $A(\alpha,\beta)+B(\alpha,\beta)>0$ for $(\alpha,\beta)\in\partial\Omega$.

5°. There exists a point $(\bar{\alpha},\bar{\beta})\in\partial\Omega$ such that $B(\bar{\alpha},\bar{\beta})>0$ or there exists a point $(\bar{\bar{\alpha}},\bar{\bar{\beta}})\in\Omega$ such that $G_r(\bar{\bar{\alpha}},\bar{\bar{\beta}},r,s,t,w,z)<0$ for $(r,s,t,w,z)\in R^5$.

Let $\Omega_h=\{(ih,kh)\colon i=1,2,\ldots,m-1,\ k=1,2,\ldots,n-1\}$, $a_1=mh$, $a_2=nh$, $h_0=\min\{M_1^{-1},M_2^{-1}\}$, $\partial\Omega_h=\{(ih,kh)\colon\ i=0,m,\ k=1,2,\ldots,n-1$ or $k=0,n,\ i=1,2,\ldots,m-1\}$.

The following difference scheme approximates the equations (5), (6):

$$G(ih,kh,v_{ik},\delta^1v_{ik},\delta^2v_{ik},L_1v_{ik},L_2v_{ik})=\varphi_{ik},\qquad(ih,kh)\in\Omega_h, \tag{7}$$

$$\begin{aligned}&\frac{A_{i0}}{h}(v_{i0}-v_{i1})+B_{i0}v_{i0}=\psi_{i0},\qquad i=1,2,\ldots,m-1,\\&\frac{A_{0k}}{h}(v_{0k}-v_{1k})+B_{0k}v_{0k}=\psi_{0k},\qquad k=1,2,\ldots,n-1,\\&\frac{A_{in}}{h}(v_{in}-v_{i-1,n})+B_{in}v_{in}=\psi_{in},\qquad i=1,2,\ldots,m-1,\\&\frac{A_{mk}}{h}(v_{mk}-v_{m,k-1})+B_{mk}v_{mk}=\psi_{mk},\qquad k=1,2,\ldots,n-1,\end{aligned} \tag{8}$$

where

$$v_{ik}=v(ih,kh),\qquad\delta^1v_{ik}=\frac{1}{2h}(v_{i+1,k}-v_{i-1,k}),\qquad\delta^2v_{ik}=\frac{1}{2h}(v_{i,k+1}-v_{i,k-1}),$$

$$L_1v_{ik}=\frac{1}{h^2}(v_{i+1,k}-2v_{ik}+v_{i-1,k}),\qquad L_2v_{ik}=\frac{1}{h^2}(v_{i,k+1}-2v_{ik}+v_{i,k-1}).$$

Now, we rewrite the difference scheme (7), (8) in the form (1). For this purpose, we introduce the following notation:

$$N = (m+1)(n+1)-4, \quad p = (p_1, p_2, \ldots, p_N), \quad p_{(m+1)k+i} = (ih, kh),$$
$$x_{(m+1)k+i} = v_{ik}, \quad I = I^1 \cup I^2 \cup I^3 \cup I^4,$$
$$I^1 = \{1, 2, \ldots, m-1\}, \quad I^2 = \{(m+1)k\colon k = 1, 2, \ldots, n-1\},$$
$$I^3 = \{(m+1)n+i\colon i = 1, 2, \ldots, m-1\},$$
$$I^4 = \{(m+1)k+m\colon k = 1, 2, \ldots, n-1\},$$

$$f^j(x) = \begin{cases} -G(p_j, x_j, \delta^1 x_j, \delta^2 x_j, L_1 x_j, L_2 x_j) & \text{for} \quad j \notin I, \\ \dfrac{A(p_j)}{h}(x_j - x_{j+m+1}) + B(p_j) x_j & \text{for} \quad j \in I^1, \\ \dfrac{A(p_j)}{h}(x_j - x_{j+1}) + B(p_j) x_j & \text{for} \quad j \in I^2, \\ \dfrac{A(p_j)}{h}(x_j - x_{j-m-1}) + B(p_j) x_j & \text{for} \quad j \in I^3, \\ \dfrac{A(p_j)}{h}(x_j - x_{j-1}) + B(p_j) x_j & \text{for} \quad j \in I^4, \end{cases}$$

$$J(M) = \{j\colon B(p_j) > 0 \text{ for } j \in I \text{ or } G_r(p_j, r, s, t, w, z) < 0 \text{ for } j \notin I\}.$$

For the vector-function $f(x)$ we have $M_{ik}(x, Q) = f^i_{x_k}(xQ)$, where

$$M_{ii} = \frac{A(p_i)}{h} + B(p_i) \quad \text{for} \quad i \in I,$$
$$M_{ii} = \frac{2}{h^2}(G^i_w + G^i_z) - G^i_r \quad \text{for} \quad i \notin I,$$
$$M_{i,i\mp 1} = -\frac{1}{h^2} G^i_w \mp \frac{1}{2h} G^i_s \quad \text{for} \quad i \notin I,$$
$$M_{i,i\mp m\mp 1} = -\frac{1}{h^2} G^i_z \mp \frac{1}{2h} G^i_t \quad \text{for} \quad i \notin I,$$
$$M_{i,i+m+1} = -\frac{A(p_i)}{h} \quad \text{for} \quad i \in I^1,$$
$$M_{i,i-m-1} = -\frac{A(p_i)}{h} \quad \text{for} \quad i \in I^3,$$
$$M_{i,i+1} = -\frac{A(p_i)}{h} \quad \text{for} \quad i \in I^2,$$
$$M_{i,i-1} = -\frac{A(p_i)}{h} \quad \text{for} \quad i \in I^4,$$
$$G^i = G(p_i, r, s, t, w, z).$$

The remaining entries of M are equal to zero.

THEOREM 2. *If there exists a point $p_l \in \partial\Omega_h$ such that $B(p_l) > 0$ or there exists a point $p_j \in \Omega_h$ such that $G_r(p_j, r, s, t, w, z) < 0$ for $(r, s, t, w, z) \in R^5$, then the matrix $M(x, Q)$ is positive type* (cf. [5]) *for every $x \in R^N$ and $Q = (Q_1, Q_2, \ldots, Q_N)$, $0 < Q_i < 1$, $i = 1, 2, \ldots, N$.*

Proof. From assumptions 1°–5° it follows that for $h \leqslant h_0$, $M_{ik} \leqslant 0$ if $i \neq k$, $\sum_{k=1}^{N} M_{ik} \geqslant 0$ for $i = 1, 2, \ldots, N$ and $\sum_{k=1}^{N} M_{ik} > 0$ if $i \in J(M)$. It is easy to verify that for $i \notin J(M)$ there exists a $k \in J(M)$ and a sequence of nonzero elements of M of the following form:

$$M_{ii_1}, M_{i_1 i_2}, \ldots, M_{i\,k}.$$

Hence, the matrix M satisfies the conditions of the definition given in [3] and the matrix M is positive type.

From Theorem 2 follows

COROLLARY 1. *If $A(\alpha, \beta) = 0$, $B(\alpha, \beta) = 1$, $\psi(\alpha, \beta) \geqslant 0$ (or $\psi(\alpha, \beta) \leqslant 0$) for $(\alpha, \beta) \in \partial\Omega$ and $\varphi(\alpha, \beta) \geqslant 0$ (or $\varphi(\alpha, \beta) \leqslant 0$) for $(\alpha, \beta) \in \Omega$, then the solution v of the difference scheme* (7), (8) *is non-negative* (*or non-positive*) *in $\Omega \cup \partial\Omega$.*

Remark 1. Theorem 2 and Corollary 1 are a difference analogue of the theorem given in [6], p. 165.

Let $K(M)$ be a non-empty subset of $J(M)$ and let S denote the subspace of R^N spanned by the vectors e_i for $i \in K(M)$.

THEOREM 3. *If $\mathring{x} \in R^N$ is a solution of $f(x) = P_S b$, then*

$$\|\mathring{x}\| \leqslant \|D^1 P_S b\|,$$

where $D^1 =$ diagonal $(D^1_1, D^1_2, \ldots, D^1_N)$,

$$D^1_i = \begin{cases} 1 & \textit{for} \quad i \notin K(M), \\ -G^i_r & \textit{for} \quad i \in K(M) - I, \\ B(p_i) & \textit{for} \quad i \in K(M) \cap I. \end{cases}$$

Proof. Let $\mathring{x}$ be a solution of $f(x) = P_S b$. There exists a $Q = (Q_1, Q_2, \ldots, Q_N)$, $0 < Q_k < 1$, $k = 1, 2, \ldots, N$ such that

$$M(\mathring{x}, Q)\mathring{x} = P_S b \quad \text{and} \quad D^1 M(\mathring{x}, Q)\mathring{x} = D^1 P_S b.$$

From Theorem 2 it follows that the matrix $D^1 M(\mathring{x}, Q) - D^2$ is positive type, where $D^2 =$ diagonal $(D^2_1, D^2_2, \ldots, D^2_N)$,

$$D^2_i = \begin{cases} 0 & \text{for} \quad i \notin J(M), \\ -G^i_r & \text{for} \quad i \in J(M) - I - K(M), \\ B(p_i) & \text{for} \quad i \in J(M) \cap I - K(M). \end{cases}$$

Furthermore, $(D^1 M - D^2)\xi = P_S \xi$. Hence $\|(D^1 M - D^2)^{-1} P_S\| \leqslant 1$ (cf. [1]). The monotone matrices $D^1 M$, $D^1 M - D^2$ satisfy the inequalities:

$$D^1 M - D^2 \leqslant D^1 M,$$
$$(D^1 M)^{-1} \leqslant (D^1 M - D^2)^{-1}.$$

Then from Corollary 1 and Lemma 1 given in [1] it follows the following inequality

$$\|\mathring{x}\| \leqslant \|D^1 P_S b\|$$

which ends the proof.

Let $\Sigma \subset \partial\Omega$ be a domain and let

$$A(\alpha,\beta) = \begin{cases}1 & \text{for} \quad (\alpha,\beta)\in\Sigma,\\ 0 & \text{for} \quad (\alpha,\beta)\in\partial\Omega-\Sigma,\end{cases}$$

$$B(\alpha,\beta) = \begin{cases}1 & \text{for} \quad (\alpha,\beta)\in\partial\Omega-\Sigma,\\ 0 & \text{for} \quad (\alpha,\beta)\in\Sigma,\end{cases}$$

$$K(M) = \{j\in I\colon\ p_j\in\partial\Omega-\Sigma\},$$

$$\psi(\alpha,\beta) = 0 \text{ for } (\alpha,\beta)\in\Sigma, \qquad \varphi(\alpha,\beta) = 0 \text{ for } (\alpha,\beta)\in\Omega.$$

From Theorem 3 follows

COROLLARY 2. *A solution v of the difference scheme* (7), (8) *satisfies the inequality*

$$\max_{\bar\Omega_h}|v_{ik}| \leqslant \max_{\partial\Omega_h-\Sigma_h}|\psi_{ik}|.$$

Remark 2. If Σ is the empty set then the system of equations (7), (8) satisfies the maximum principle in the sense of Definition 1.

THEOREM 4. *If $A(\alpha,\beta) = 0$, $B(\alpha,\beta) = 1$ for $(\alpha,\beta)\in\partial\Omega$, then the solution v of the difference scheme* (7), (8) *satisfies the following inequality*:

$$\max_{\bar\Omega_h}|v_{ik}| \leqslant \max_{\partial\Omega_h}|\psi_{ik}|+\exp(\gamma)\max_{\Omega_h}|\varphi_{ik}|, \tag{9}$$

where

$$\gamma = \min\left\{\frac{K+\sqrt{K^2+2\mu}}{\mu}, \frac{L+\sqrt{L^2+2\nu}}{\nu}\right\}.$$

Proof. If $x_{(m+1)k+i} = v_{ik}$ then x is a solution of the following system of equations:

$$M(x,Q)x = b \quad \text{for a certain } Q, \tag{10}$$

where

$$b_{(m+1)k+i} = -\varphi(ih,kh) \quad \text{for} \quad (ih,kh)\in\Omega_h,$$
$$b_{(m+1)k+i} = \psi(ih,kh) \quad \text{for} \quad (ih,kh)\in\partial\Omega_h.$$

Now, we notice that the assumptions of Theorem 1 are satisfied for $\hat\alpha = (\alpha_1,\alpha_2,\ldots$ $\ldots,\alpha_N)^*$, where

$$\alpha_j = \begin{cases}\exp(\gamma)-(1-\gamma h)^j, & j = 1,2,\ldots,m,\\ \alpha_{j-m}, & j = m+1,m+2,\ldots,N,\end{cases} \qquad \gamma = \frac{K+\sqrt{K^2+2\mu}}{\mu}.$$

Setting $x_{(m+1)i+k} = v_{ik}$, we can take $\gamma = \big(L+\sqrt{L^2+2\nu}\big)/\nu$.

3. A maximum principle for a system of differential equations

Let us consider the following initial value problem:

$$Pv \equiv \frac{dv}{dt}-A(t,v)v = b(t), \qquad 0<t<T, \tag{11}$$

$$v(0) = c, \tag{12}$$

where $v = (v^1,v^2,\ldots,v^N)^*$, $b = (b^1,b^2,\ldots,b^N)^*$, $c = (c^1,c^2,\ldots,c^N)^*$, $A = (A_{ij})$, $i,j = 1,2,\ldots,N$.

We assume that the entries of the given matrix $A(t,v)$ and the vector $b(t)$ are continuous in $\langle 0,T\rangle$.

DEFINITION 3. An operator P satisfies the *maximum principle with respect to the subspace* S (in symbols, $P\in\mathfrak{M}_S^t$) if and only if every solution $v(t)$ of the problem

$$\frac{dv}{dt} = A(t,v)v+P_Sb \quad \text{for any } b\in C(\langle 0,T\rangle),\ 0<t<T, \tag{13}$$

$$v(0) = c, \tag{14}$$

satisfies the inequality

$$\|v\| \leqslant \max\{\|P_Sb\|+\|c\|, \|P_Sv\|\}, \tag{15}$$

where $\|v\| = \max\limits_{1\leqslant i\leqslant N}\max\limits_{0\leqslant t\leqslant T}|v^i(t)|$.

THEOREM 5. *If an $N\times N$ matrix $A(t,v)$ is positive type and*

$$\sum_{j=1}^N A_{ij}(t,v) = 1 \quad \textit{for} \quad e_i\in S,$$

then $P\in\mathfrak{M}_S^t$.

Proof. There exist integers $j,k\in\{1,2,\ldots,N\}$ such that

$$\max_{1\leqslant i\leqslant N}\max_{0\leqslant t\leqslant T} v^i(t) = v^j(t_0),$$
$$\min_{1\leqslant i\leqslant N}\min_{0\leqslant t\leqslant T} v^i(t) = v^k(t_1).$$

If $t_0, t_1\in(0,T)$ then from (13) it follows that

$$\sum_{i=1}^N A_{ji}\big(t_0,v(t_0)\big)v^i(t_0) = \begin{cases}0 & \text{for} \quad j\notin I,\\ -b^j(t_0) & \text{for} \quad j\in I,\end{cases} \tag{16}$$

and

$$\sum_{i=1}^N A_{ki}\big(t_1,v(t_1)\big)v^i(t_1) = \begin{cases}0 & \text{for} \quad k\notin I,\\ -b^k(t_1) & \text{for} \quad k\in I,\end{cases} \tag{17}$$

where $I = \{i\colon\ e_i\in S\}$.

Since the matrix $A(t,v)$ is positive type, then for $v^j(t_0)>0$ and $v^k(t_1)<0$, we have

$$v^j(t_0)\sum_{i=1}^N A_{ji}(t_0,v) < \begin{cases}0 & \text{for} \quad j\notin I,\\ -b^j(t_0) & \text{for} \quad j\in I,\end{cases} \tag{18}$$

$$v^k(t_1)\sum_{i=1}^N A_{ki}(t_1,v) > \begin{cases}0 & \text{for} \quad k\notin I,\\ -b^k(t_1) & \text{for} \quad k\in I.\end{cases} \tag{19}$$

If $j,k\notin I$ then inequalities (18) and (19) are contradictory to (16) and (17). From (18) and (19) it follows that

$$v^i(t) \leqslant -b^j(t_0) \leqslant \|P_Sb\| \quad \text{for} \quad i = 1,2,\ldots,N,\ t\in(0,T),$$

and

$$v^i(t) \geqslant -b^k(t_1) \geqslant -\|P_S b\| \quad \text{for} \quad i = 1, 2, \ldots, N, \; t \in (0, T).$$

Hence, we have inequality (15) and this ends the proof.

DEFINITION 4. An operator P satisfies the *maximum principle in the sense an a priori estimation with respect to the subspace S and a number K* (in symbols, $P \in \mathfrak{M}_S^t(K)$) if and only if every solution $v(t)$ of (11), (12) satisfies inequality

$$\|v\| \leqslant \|c\| + \|P_S b\| + \|P_S v\| + K\|(E-P_S)b\| \quad \text{for any } b \in C(\langle 0, T\rangle). \tag{20}$$

THEOREM 6. *If an $N\times N$ uniformly monotone matrix $A(t, v)$ satisfies the inequality*

$$A(t, v)\xi \geqslant P_S \xi \quad \textit{for} \quad t \in (0, T), \; \xi = (1, 1, \ldots, 1)^*$$

and if there exists a vector $\hat{\alpha} \in R^N$ such that $\|\hat{\alpha}\| \leqslant K$ and

$$A(t, v)\,\hat{\alpha} \geqslant (E-P_S)\xi,$$

then $P \in \mathfrak{M}_S^t(K)$.

Proof. Let $v(t)$ be a solution of (11), (12) and let

$$g = \|P_S b\|\xi + \|(E-P_S)b\|\hat{\alpha}.$$

From (11) we have

$$A(g \mp v) = Ag \mp \frac{dv}{dt} \pm b = \|P_S b\| A\xi + \|(E-P_S)b\| A\hat{\alpha} \mp \frac{dv}{dt} \pm b. \tag{21}$$

Now, suppose that at some point $t_i \in (0, T)$

$$\max_{0 \leqslant t \leqslant T} v^i(t) = v^i(t_i).$$

From (21) we obtain

$$A\big(g(t_i) \mp v(t_i)\big) \geqslant \|P_S b\| P_S \xi + \|(E-P_S)b\|(E-P_S)\xi \pm b(t_i) \geqslant 0.$$

Since the matrix $A(t, v)$ is monotone then $g \mp v \geqslant 0$, and

$$\|v\| \leqslant \|P_S b\| + K\|(E-P_S)b\|$$

and this ends the proof.

Remark 3. If an operator $P \in \mathfrak{M}_S^t(K)$ then the operator $\overline{P} \in \mathfrak{M}_S^t(K)$, where $Pv \equiv \dfrac{dv}{dt} - A(t, v)v$, $\overline{P}v \equiv \dfrac{dv}{dt} - \overline{A}(t, v)$, $\overline{A}(t, v) = \big(\overline{A}_{ij}(t, v)\big)$, $i, j = 1, 2, \ldots, N$, $\overline{A}_{ij}(t, v) = A_{ij}(t, v)$ for $e_i \notin S$, $\overline{A}_{ij}(t, v) = -A_{ij}(t, v)$ for $e_i \in S$.

4. A method of lines for nonlinear parabolic equations

In the papers [7] and [8] is given an estimation of an error of a method of lines for nonlinear parabolic equations. Below, we present different estimation of this error also for nonlinear parabolic equations.

In the domain $Q_T = \{(\alpha, t)\colon 0 < \alpha < 1, \; 0 < t < T\}$ we consider the following problem:

$$\frac{\partial u}{\partial t} = G(\alpha, t, u, u_\alpha, u_{\alpha\alpha}) + g(\alpha, t), \quad (\alpha, t) \in Q_T, \tag{22}$$

$$u(0, t) = u(1, t) = 0, \quad 0 \leqslant t \leqslant T, \tag{23}$$

$$u(\alpha, 0) = \psi(\alpha), \quad 0 \leqslant \alpha \leqslant 1. \tag{24}$$

ASSUMPTIONS. $1^{\circ\circ}$. The given function $G(\alpha, t, p, r, s)$ is continuous in $Q_T \times R^3$ and differentiable with respect to variables p, r, s, $G(\alpha, t, 0, 0, 0) = 0$ for $(\alpha, t) \in Q_T$. There exist functions $\mu(p, r, s) > 0$, $K(p, r, s)$, $L(p, r, s)$ and a constant M such that

$$G_s(\alpha, t, p, r, s) \geqslant \mu(p, r, s), \tag{25}$$

$$G_p(\alpha, t, p, r, s) \leqslant 0, \tag{26}$$

$$|G_r(\alpha, t, p, r, s)| \leqslant K(p, r, s), \tag{27}$$

$$K(p, r, s) \leqslant M\mu(p, r, s) \tag{28}$$

for $(\alpha, t, p, r, s) \in Q_T \times R^3$. The functions $g(\alpha, t)$, $\psi(\alpha)$ are continuous in $\overline{Q}_T$ and $0 \leqslant \alpha \leqslant 1$, respectively.

The following system of ordinary differential equations approximates the problem (22)–(24):

$$\frac{dv}{dt} = v^0,$$

$$\frac{dv^i}{dt} = G(ih, t, v^i, \delta^1 v^i, L_1 v^i) + g(ih, t), \quad i = 1, 2, \ldots, N-1, \tag{29}$$

$$v_0^0 = v_0^N = 0, \quad v_0^i = \psi(ih), \quad i = 1, 2, \ldots, N-1, \tag{30}$$

where

$$Nh = 1, \quad v = (v^0, v^1, \ldots, v^N)^*, \quad v^i = v(t, ih), \quad i = 0, 1, \ldots, N,$$

$$\delta^1 v^i = \frac{1}{2h}(v^{i+1} - v^{i-1}), \quad L_1 v^i = \frac{1}{h^2}(v^{i-1} - 2v^i + v^{i+1}).$$

From assumptions $1^{\circ\circ}$ it follows that

$$G(ih, t, v^i, \delta^1 v^i, L_1 v^i) = \sum_{k=0}^{N} \frac{\partial G(ih, t, \theta_i v^i, \theta_i \delta^1 v^i, \theta_i L_1 v^i)}{\partial v^k} v^k, \tag{31}$$

for certain $\theta = (\theta_0, \theta_1, \ldots, \theta_N)$.

Let $A_{ii} = 1$ for $i = 0, N$, $A_{ii} = \dfrac{\partial G}{\partial v^i}$, $A_{ii\mp 1} = \dfrac{\partial G}{\partial v^{i\mp 1}}$ for $i = 1, 2, \ldots, N-1$, the remaining entries of the matrix A are equal to zero. Now, we rewrite the system (29), (30) in the following form:

$$\frac{dv}{dt} = Av + b, \quad 0 < t < T, \tag{32}$$

$$v(0) = c, \tag{33}$$

where $b = (b^0, b^1, \ldots, b^N)^*$, $b^0 = b^N = 0$, $b^i = g(ih, t)$, $i = 1, 2, \ldots, N-1$, $c = (0, \psi(h), \psi(2h), \ldots, \psi(1))^*$.

Let $u(\alpha, t)$ be a solution of (22)–(24) and let $v(t)$ be a solution of (32), (33). From (22) and (24) it follows that

$$\frac{d\vec{u}}{dt} = A\vec{u}+b+\omega(h), \quad 0 < t < T, \tag{34}$$

$$\vec{u}(0) = c, \tag{35}$$

where $\vec{u}(t) = \big(u(0, t), u(h, t), \ldots, u(1, t)\big)^*$, $\omega(h) \to 0$ as fast as $h^2 \to 0$.

From (32), (33) and (34), (35) it follows that:

$$\frac{d(\vec{u}-v)}{dt} = A(\vec{u}-v)+\omega(h), \quad 0 < t < T,$$

$$\vec{u}(0)-v(0) = 0.$$

Now, we notice that the matrix A and the vector $\hat{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$, $\alpha_j = \exp(\gamma)-(1-\gamma h)^j$, $j = 0, 1, \ldots, N$, $\gamma = \big(K+\sqrt{K^2+2\mu}\big)/\mu$ satisfy the assumptions of Theorem 6. Therefore we have the inequality $\|\vec{u}-v\| \leqslant K_0 h^2$, where $K_0 = \text{const}$ for $\mu(p, r, s) \geqslant \mu_0 > 0$.

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ОБ УСТОЙЧИВОСТИ ОТНОСИТЕЛЬНО ИЗМЕНЕНИЙ КРАЯ ОДНОЙ РАЗНОСТНОЙ СХЕМЫ

В. ВАЙНЕЛЬТ

Технический БУЗ, Карл-Маркс-Штадт, ГДР

1

Устойчивость разностных схем исследована относительно многих различных ситуаций, например, исследованы устойчивость по правой части, по начальным данным (если задача нестационарна), по краевым данным и по изменениям коэффициентов.

Однако, мало изучен вопрос о том, как изменяется разностное решение, если меняется граница области краевой задачи. Ответ на такой вопрос актуален, например, в следующих случаях:

(а) Уже при моделировании реальной системы часто область краевой задачи задана неточно из-за неточности измерений или по причинам удобного представления задачи.

(б) Для того, чтобы применить хороший вычислительный метод, иногда упрощается область задачи.

(в) Имеются задачи, при которых граница частично свободна. При этом нужны схемы, обладающие свойством устойчивости по изменениям границы.

Здесь приводится простой случай одномерной краевой задачи.

2

Для краевой задачи для уравнения Штурма–Лиувилля:

$$-(p(x)u')'+q(x)u = f(x) \quad (x \in (0, l))$$

с краевыми условиями либо третьего рода:

$$\alpha_1 u(0)-p(0)u'(0) = \gamma_1,$$
$$\alpha_2 u(l)+p(l)u'(l) = \gamma_2$$