APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

OPERATOR NORM BOUNDS AND ERROR BOUNDS FOR QUADRATIC SPLINE INTERPOLATION

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INTRODUCTION

The following work began in 1971 as a result of a conversation with A. Sharma in which [19] by Subbotin was brought to my attention.

In Part I proofs will be given of results which were announced in [13]. These proofs are elementary — they generally require at most the triangle inequality or Taylor's theorem with derivative remainder. However, I supply complete details since there may be some interest in the constants in the error bounds. See [7], p. 118. It may be that quadratic spline interpolation is preferable to cubic spline interpolation when C^2 or C^3 functions are considered and when error is measured by the uniform norm.

In Part II I will discuss, following de Boor [3], the exponential growth of null splines. However, these null splines vanish between knots rather than at the knots. They seem to be connected with "midpoint" Euler–Frobenius polynomials. See [16], Lecture 3, Sec. 4. Here I use only the quadratic null splines to provide a second proof that quadratic spline interpolation is operator norm bounded with no knot restrictions. Their properties (see also [3]) suggest another proof of Theorems 5.1 and 5.3 in Kammerer, Reddien, and Varga [9] concerning local error bounds for interpolation to locally smooth functions. I give only the proof of Theorem 5.1.

Finally, in Part III I will define quadratic B-splines and provide a third proof that the quadratic spline interpolation operators are uniformly bounded. A fourth proof has been given by de Boor [2], Sec. 3.

In Part I the proofs are not very interesting. They usually involve the mean-value theorem and the triangle inequality. I am sure that some of the arguments may be tightened to produce smaller constants. The dependence on h can be made "local". See [9], Th. 4.3. In Part II the result is not interesting, but (I think) the technique is interesting. In future work I hope to use null-splines to produce a simpler proof of de Boor's result in [2] that cubic spline interpolation at successive three-knot averages is operator norm bounded with no knot restrictions. The diagonal dominance argument used in Part III cannot be used for higher degree splines. In [2], de Boor has exploited *total positivity* to produce the cubic spline result, already mentioned. An extension of his lemma ([5], p. 457) may be needed before the approach of [2] or of Part III can be used on splines of degree four or more.

It should be noted that (d) below in the definition of quadratic spline interpolation differs from an approach sometimes used (and preferred, see [6]) for even-degree spline interpolation wherein knots are placed midway between successive pairs of interpolation nodes. My choice of (d) was suggested by Schoenberg's use ([17], [14], [12]) of the nodes

$$\tau_{j,k+1} = (x_{j+1}+x_{j+2}+\ldots+x_{j+k})/k$$

in defining variation diminishing spline approximation of degree k. In this connection, Theorem 1 below should be contrasted with a result of Zmatrakov [20]. Recently, de Boor [2] has proved that the cubic spline interpolation operators with the nodes $\tau_{j,4}$ are uniformly bounded. Again see [20] for a contrasting result when the knots x_j are used as nodes for cubic spline interpolation.

The assumption of periodicity, (c) in the definition of quadratic spline interpolation, can be replaced by other boundary conditions, e.g. $s(0) = y(0)$ and $s(1) = y(1)$. When this is done, the assumption of periodicity on y may be relaxed. In this case, it is convenient to think of s as a periodic quadratic spline having triple knots (and, hence, one-sided continuity) at the integers. The only modification necessary is that the consistency relations (1) and (2) do not hold for $i = n$.

DEFINITIONS AND PRELIMINARY RESULTS

Let $\Delta\colon 0 = x_0 < x_1 < \ldots < x_n = 1$ be a (fixed) partition of $[0, 1]$. Set $h_i = x_i - x_{i-1}$ and $\tau_i = (x_{i-1}+x_i)/2$ for $i = 1, \ldots, n$. Set $h_0 = h_n$ and $h_{n+1} = h_1$. Set $a_i = h_{i+1}/(h_i+h_{i+1})$ and $c_i = 1-a_i$ for $i = 1, \ldots, n$. For any function g, set $g_i = g(x_i)$ for $i = 0, 1, \ldots, n$. For any n-vector (v_i), let $\|v_i\| = \max\{|v_i|\colon i = 1, \ldots, n\}$. For any function g, let

$$\|g\| = \|g\|_{L_\infty[0,1]} = \sup\{|g(x)|\colon 0 \leqslant x \leqslant 1\}.$$

Let $y \in L_\infty[0, 1]$, the space of bounded functions on $[0, 1]$ with the above norm. We shall usually assume that $y(0) = y(1)$ and that y is extended periodically with period 1. A function $s = P_\Delta y$ is a *periodic quadratic spline interpolant* associated with y and Δ if

(a) $s(x)$ is a quadratic expression on each (x_{i-1}, x_i);

(b) $s \in C^1[0, 1]$, the space of differentiable functions;

(c) $s(0) = s(1)$, $s'(0) = s'(1)$; and

(d) $s(\tau_i) = y(\tau_i)$ for $i = 1, \ldots, n$.

Thus, s interpolates y at the midpoints of the subintervals formed by Δ, the set of knots of s.

On (x_{i-1}, x_i),

$$s(x) = [s_i'(x-x_{i-1})^2 - s_{i-1}'(x_i-x)^2]/(2h_i) + s(\tau_i) - h_i(s_i'-s_{i-1}')/8$$

so that

$$s(x_i) = s(x_i-) = s(\tau_i) + (3s_i'+s_{i-1}')h_i/8$$

and

$$s(x_{i-1}) = s(x_{i-1}+) = s(\tau_i) - (3s_{i-1}'+s_i')h_i/8.$$

Equating $s(x_i-)$ and $s(x_i+)$ as required by (b) yields

$$c_i s_{i-1}' + 3s_i' + a_i s_{i+1}' = 8[s(\tau_{i+1}) - s(\tau_i)]/(h_i+h_{i+1}) \tag{1}$$

for $i = 1, \ldots, n$.

Similarly, since on (x_{i-1}, x_i)

$$h_i^2 s(x) = 4s(\tau_i)(x_i-x)(x-x_{i-1}) + [s_{i-1}(x_i-x) - s_i(x-x_{i-1})](x_{i-1}+x_i-2x),$$

we have

$$s'(x_i-) = (s_{i-1} - 4s(\tau_i) + 3s_i)/h_i$$

and

$$s'(x_{i-1}+) = (-3s_{i-1} + 4s(\tau_i) - s_i)/h_i$$

so that

$$a_i s_{i-1} + 3s_i + c_i s_{i+1} = 4a_i s(\tau_i) + 4c_i s(\tau_{i+1}) \tag{2}$$

for $i = 1, \ldots, n$ follows from the relation $s'(x_i-) = s'(x_i+)$.

In view of (d), the right members of (1) and (2) are "known". Either (1) or (2) may be solved, since the coefficient matrices are diagonally dominant. Thus, the questions of existence and uniqueness are settled.

Set $e = y - s = (I-P_\Delta)y$. Since $e_i = y_i - s_i$ and $e_i' = y_i' - s_i'$ (when y_i' exists), the relations

$$a_i e_{i-1} + 3e_i + c_i e_{i+1} = a_i y_{i-1} + 3y_i + c_i y_{i+1} - 4a_i y(\tau_i) - 4c_i y(\tau_{i+1}) \tag{3}$$

and

$$c_i e_{i-1}' + 3e_i' + a_i e_{i+1}' = c_i y_{i-1}' + 3y_i' + a_i y_{i+1}' - 8[y(\tau_{i+1}) - y(\tau_i)]/(h_i+h_{i+1}) \tag{4}$$

for $i = 1, \ldots, n$ follow readily from (1), (2), and (d).

The following preliminary bounds on s_i, s_i', e_i, e_i' are basic to Theorems 1–4:

$$\|s_i\| \leqslant 2\|y(\tau_i)\|; \tag{5}$$

$$\|s_i'\| \leqslant 4\|[y(\tau_{i+1}) - y(\tau_i)]/(h_i+h_{i+1})\|; \tag{6}$$

$$2\|e_i\| \leqslant \|a_i y_{i-1} + 3y_i + c_i y_{i+1} - 4a_i y(\tau_i) - 4c_i y(\tau_{i+1})\|; \tag{7}$$

and

$$2\|e_i'\| \leqslant \|c_i y_{i-1}' + 3y_i' + a_i y_{i+1}' - 8[y(\tau_{i+1}) - y(\tau_i)]/(h_i+h_{i+1})\|. \tag{8}$$

The following derivation of (7) emphasizes the "local" character of our argument. See Sharma and Meir [18] where the same approach was used to produce cubic spline interpolation error bounds.

For a fixed j,

$$\begin{aligned}3|e_j|-||e_i|| &\leqslant 3|e_j|-\max\{|e_{j-1}|,\,|e_{j+1}|\}\\ &\leqslant 3|e_j|-a_j|e_{j-1}|-c_j|e_{j+1}| \leqslant |3e_j+a_je_{j-1}+c_je_{j+1}|\\ &= |a_jy_{j-1}+3y_j+c_jy_{j+1}-4a_jy(\tau_j)-4c_jy(\tau_{j+1})|\\ &\leqslant ||a_iy_{i-1}+3y_i+c_iy_{i+1}-4a_iy(\tau_i)-4c_iy(\tau_{i+1})||.\end{aligned}$$

Now, take the supremum to get (7).

A similar argument, using (2) instead of (3), produces

$$3|s_j|-||s_i|| \leqslant |4a_js(\tau_j)+4c_js(\tau_{j+1})| \leqslant 4(a_j+c_j)||s(\tau_i)|| = 4||y(\tau_i)||$$

from which (5) follows.

We now give a second derivation of (7), using a well-known matrix bound:

$$||A^{-1}|| \leqslant ||D||/(1-||I-DA||) \quad \text{if} \quad ||I-DA|| < 1.$$

See Hall [8] for the use of this approach to produce cubic spline interpolation error bounds.

Let A be the $n\times n$ coefficient matrix on the left of (3). Let k_i denote the right members of (3). Thus, (3) reads $A(e_j) = (k_i)$. Then

$$\begin{aligned}||e_i|| = ||A^{-1}(k_i)|| &= \max_i\Big|\sum_j (A^{-1})_{ij}k_j\Big| \leqslant \max_i\sum_j |(A^{-1})_{ij}|\cdot|k_j|\\ &\leqslant ||k_i||\max_i\sum_j |(A^{-1})_{ij}| = ||k_i||\cdot||A^{-1}||\\ &\leqslant ||k_i||\cdot||D||/(1-||I-DA||) \leqslant (1/2)||k|| \quad \text{if} \quad D = (1/3)I.\end{aligned}$$

This result is equivalent to (7).

The relations (6) and (8) can be derived by either of the two approaches just demonstrated.

PART I

I.1. Statement of the main results

Recall that

$$\omega(y;\delta) = \sup\{|y(x')-y(x'')|:\ 0 \leqslant x'-x'' \leqslant \delta\}.$$

Since y is 1-periodic, we may restrict x' and x'' to the interval $[-\delta, 1]$. Recall also that

$$||P_\Delta|| = \sup\{||P_\Delta y||:\ ||y|| = 1\}.$$

THEOREM 1. *Let Δ be a partition of* $[0, 1]$, *y be a bounded* 1-*periodic function, and $s = P_\Delta y$ be the periodic quadratic spline interpolant associated with y and Δ. Then*

(9) $$||s_i|| \leqslant 2||y||, \quad ||s|| \leqslant 2||y||, \quad ||P_\Delta|| \leqslant 2;$$

(10) $$||e_i|| \leqslant 2\omega(y;h/2);$$

and

(11) $$||e|| \leqslant 3\omega(y;h/2) \quad \textit{where} \quad h = ||h_i||.$$

The constant 2 which appears in (9) *cannot, in general, be decreased.*

THEOREM 2. *Let y and y' be continuous* 1-*periodic functions. Then*

(12) $$||s'|| = ||s_i'|| \leqslant 2||y'||, \quad ||e'|| \leqslant 3||y'||;$$

(13) $$||e_i|| \leqslant (1/2)h\omega(y';h) \leqslant h\omega(y';h/2), \quad ||e_i|| \leqslant h||y'||;$$

(14) $$||e|| \leqslant (5/4)h||y'||;$$

(15) $$||e_i'|| \leqslant 3\omega(y';h/2);$$

(16) $$||e'|| \leqslant (9/2)\omega(y';h/2);$$

and

(17) $$||e|| \leqslant (13/8)h\omega(y';h/2).$$

THEOREM 3. *Let y, y', and y'' be continuous* 1-*periodic functions. Then*

(18) $$||e_i|| \leqslant (1/8)h^2\omega(y'';h);$$

(19) $$||e_i'|| \leqslant (1/2)h\omega(y'';h);$$

(20) $$||e'|| \leqslant h\omega(y'';h) \leqslant 2h\omega(y'';h/2), \quad ||e'|| \leqslant 2h||y''||;$$

(21) $$||e|| \leqslant (5/16)h^2\omega(y'';h) \leqslant (5/8)h^2\omega(y'';h/2), \quad ||e|| \leqslant (5/8)h^2||y''||;$$

and

(22) $$|e''(x)| \leqslant (1+h/h_i)\omega(y'';h) \quad \textit{for} \quad x_{i-1} < x < x_i.$$

THEOREM 4. *Let y, y', y'', and y''' be continuous* 1-*periodic functions. Then*

(23) $$||e_i|| \leqslant (1/8)h^3||y'''||;$$

(24) $$||e_i'|| \leqslant (1/3)h^2||y'''||;$$

(25) $$||e'|| \leqslant (11/24)h^2||y'''||;$$

(26) $$||e|| \leqslant (17/96)h^3||y'''||;$$

and

(27) $$|e''(x)| \leqslant [h_i+(2/3)h^2/h_i]\cdot||y'''|| \quad \textit{for} \quad x_{i-1} < x < x_i.$$

Except for (22) and (27) from Kammerer, Reddien, and Varga [9], which have been added here for completeness, these results were announced in [13].

I.2. Proof of Theorem 1

Since $||y(\tau_i)|| \leqslant ||y||$, the first relation in (9) follows from (5). From this relation and convexity,

$$|s_{i-1}(x_i-x)-s_i(x-x_{i-1})| \leqslant 2h_i||y|| \quad \text{for} \quad x_{i-1} \leqslant x \leqslant x_i$$

so that, on $[x_{i-1}, x_i]$,

$$s(x) = [s_{i-1}(x_i-x)-s_i(x-x_{i-1})](x_{i-1}+x_i-2x)/h_i^2+4y(\tau_i)(x_i-x)(x-x_{i-1})/h_i^2$$

is bounded by

$$2h_i\|y\|\cdot|x_{i-1}+x_i-2x|/h_i^2+4\|y\|(x_i-x)(x-x_{i-1})/h_i^2$$

which is maximized at $x = x_i$ with a maximum value of $2\|y\|$. Thus, $\|s\| \leqslant 2\|y\|$. Recalling that $s = P\ y$ and the definition of $\|P\ \|$ completes the proof of (9).

To derive (10), we recall that $a_i+c_i = 1$, $a_i > 0$, $c_i > 0$ and rewrite (7) as

$$(28)\quad 2\|e_i\| \leqslant \|c_i[y_{i+1}-y(\tau_{i+1})]-3c_i[y(\tau_{i+1})-y_i]+3a_i[y_i-y(\tau_i)]-a_i[y(\tau_i)-y_{i-1}]\|$$

which implies that

$$2\|e_i\| \leqslant \|c_i+3c_i+3a_i+a_i\|\omega(y; h/2) = 4\omega(y; h/2).$$

We use (10) to derive (11). Let x be such that $|e(x)| = \|e\|$. Set $\omega = \omega(y; h/2)$. Suppose without loss of generality that $\tau_i \leqslant x \leqslant x_i$ for some i. If s is monotone, say, increasing, on (τ_i, x_i), then

$$e(x) = y(x)-s(x) \geqslant [y(x)-y(x_i)]+[y(x_i)-s(x_i)] \geqslant -\omega-2\omega = -3\omega$$

and

$$e(x) = y(x)-s(x) \leqslant [y(x)-y(\tau_i)]+[y(\tau_i)-s(\tau_i)] = y(x)-y(\tau_i) \leqslant \omega$$

so that $|e(x)| \leqslant 3\omega$. Similarly, if s is decreasing on (τ_i, x_i), then $|e(x)| \leqslant 3\omega$. If s is not monotone on (τ_i, x_i), the analysis is more difficult. Without loss of generality, assume that $y(\tau_i) = s(\tau_i) = 0$ and that $s_{i-1} > 0$. Then s is a parabola and $s'(x') = 0$ for some x' in (τ_i, x_i). Now, properties of parabolas and (10) give that $s(x) \geqslant s(x')$ and

$$-3s(x') \leqslant s(x_{i-1}) \leqslant 2\omega+y(x_{i-1}) \leqslant 3\omega.$$

Thus, $-s(x') \leqslant \omega$. With this fact,

$$e(x) = y(x)-s(x) \leqslant [y(x)-y(x')]+[y(x')]-[s(x')] \leqslant \omega+\omega+\omega = 3\omega$$

while, also,

$$\begin{aligned} e(x) \geqslant y(x)-\max\{s(\tau_i), s(x_i)\} &= [y(x)-y(\tau_i)]+[y(\tau_i)-\max\{y(\tau_i), s(x_i)\}] \\ &= [y(x)-y(x_i)]+[y(x_i)-\max\{y(\tau_i), s(x_i)\}] \geqslant -\omega-2\omega = -3\omega. \end{aligned}$$

Here, the last inequality follows from at least one of the two lines which precede it. The proof of (11) is complete.

To complete the proof of Theorem 1 we must show that

$$\sup\{\|P_\Delta\|: \Delta \text{ a partition of } [0, 1]\} = 2.$$

Thus, for $\varepsilon > 0$, we must show that there is a Δ and a y for which $\|y\| = 1$ while $\|P_\Delta y\| > 2-\varepsilon$. We will exhibit such a y and Δ.

We note that the approach used by Oskolkov in [15] can be paraphrased to yield an alternate proof.

Let $\varepsilon > 0$. Set $m = \max\{2, 4/\varepsilon\}$. Set k equal to an integer greater than $\log_3(3m/2)$. Then

$$2-\varepsilon < 2(m-1)/(m+1) \quad\text{and}\quad 3m/2 < 3^k.$$

Set $n = 2k+1$ and $h_j = h_{n+1-j} = m^{j-1}/(2+2m+\ldots+2m^{k-1}+m^k)$, for $j = 1, \ldots, k+1$. Define Δ by setting $x_i = h_1+\ldots+h_i$ for $i = 1, \ldots, n$. It is now claimed that $\|P_\Delta\| > 2-\varepsilon$. To prove this we construct y with $\|y\| = 1$ and show that $2-\varepsilon < P_\Delta y(x_k) = s_k \leqslant \|P_\Delta y\|$.

With $\tau_i = (x_{i-1}+x_i)/2$ as usual, set $y(\tau_{k+1}) = 0$ and $y(\tau_{k+1+j}) = -y(\tau_{k+1-j}) = (-1)^j$ for $j = 1, \ldots, k$. Complete the definition of y in any convenient way subject to the constraints $\|y\| = 1$ and $y(0) = y(1)$. The relations (2) become, since symmetry gives $s_j = -s_{2k+1-j}$ for $j = 0, 1, \ldots, k$,

$$s_0 = 0;$$

$$ms_{j-1}+3(1+m)s_j+s_{j+1} = (-1)^{k-j}4(m-1) \quad\text{for}\quad j = 1, \ldots, k-1; \quad\text{and}$$

$$ms_{k-1}+(2+3m)s_k = 4m.$$

The general solution of the middle set of difference equations is

$$s_j = [2(m-1)/(m+1)](-1)^{k-j}+B(-r_1)^j+C(-r_2)^j \quad\text{for}\quad j = 0, 1, \ldots, k$$

where $r_1 > r_2 > 0$ satisfy $r^2-3(1+m)r+m = 0$. Imposing the conditions $s_0 = 0$ and $ms_{k-1}+(2+3m)s_k = 4m$, as we must, yields

$$B+C = (-1)^{k-1}2(m-1)/(m+1)$$

and

$$B(-r_1)^k(r_1-1)+C(-r_2)^k(r_2-1) = 4.$$

These solve as

$$B = [4-r_2^k2(m-1)(1-r_2)/(m+1)]/(-1)^kD,$$

$$C = [-4-r_1^k2(m-1)(r_1-1)/(m+1)]/(-1)^kD$$

where

$$D = r_1^k(r_1-1)+r_2^k(1-r_2).$$

Thus,

$$\begin{aligned} s_k &= [2(m-1)/(m+1)]+[4(r_1^k-r_2^k)-2m^k(m-1)(r_1-r_2)/(m+1)]/D \\ &> [2(m-1)/(m+1)]+[4(r_1^k-r_2^k)-6m^{k+1}]/D \end{aligned}$$

since $D > 0$ and $r_1-r_2 < 3(m+1)$. Since $r_1 > 3m+2$ and $r_2 < 2$, it follows that $r_1^k-r_2^k > (3m)^k$ so that we have

$$s_k > [2(m-1)/(m+1)]+4m^k(3^k-3m/2)/D > 2(m-1)/(m+1) > 2-\varepsilon$$

in view of our choice of k and m.

The proof of Theorem 1 is complete.

The constants which appear in (10) and (11) may not be the "best". Simple examples with $n = 3$ can be used to establish that, in general, the constant in (10) must be at least 5/3 and the constant in (11) must be at least 2.

I.3. Proof of Theorem 2

Since s' is a piecewise linear function with corners (x_i, s_i'), we have $\|s'\| = \|s_i'\|$. Now (6) and the mean-value theorem yield

$$\|s'\| \leqslant 4\|[y(\tau_{i+1})-y(\tau_i)]/(h_i+h_{i+1})\| \leqslant 2\|y'\|.$$

Then,

$$\|e'\| = \|y'-s'\| \leqslant \|y'\|+\|s'\| \leqslant 3\|y'\|,$$

completing the proof of (12).

From (7) in the equivalent form given by (28) and the mean-value theorem,

$$\begin{aligned} 2\|e_i\| &\leqslant \|\tfrac{1}{2}c_i h_{i+1}[y'(u_4)-3y'(u_3)]+\tfrac{1}{2}a_i h_i[3y'(u_2)-y'(u_1)]\| \\ &= \|\tfrac{1}{2}a_i h_i\{[y'(u_4)-y'(u_3)]-2[y'(u_3)-y'(u_2)]+[y'(u_2)-y'(u_1)]\}\| \end{aligned}$$

where $x_{i-1} < u_1 < \tau_i < u_2 < x_i < u_3 < \tau_{i+1} < u_4 < x_{i+1}$. We have used the relation $a_i h_i = c_i h_{i+1}$. Of course, u_1, u_2, u_3, u_4 depend on i, but we have suppressed that dependence. Since $\frac{1}{2}a_i h_i = h_i h_{i+1}/(2h_i+2h_{i+1}) \leqslant h/4$, it follows that $2\|e_i\| \leqslant (h/4)\cdot 4\omega(y'; h)$. The other relations in (13) follow from the facts $\omega(g; 2\delta) \leqslant 2\omega(g; \delta)$ and $\omega(g; \delta) \leqslant 2\|g\|$.

From (12) and the relation

$$e(x) = \int_{\tau_i}^{x} e'(t)\,dt, \tag{29}$$

we have

$$|e(x)| \leqslant |x-\tau_i|\cdot\|e'\| \leqslant 3|x-\tau_i|\cdot\|y'\| \leqslant (5/4)h\|y'\|$$

provided that $|x-\tau_i| \leqslant 5h/12$. From (12) and the relation

$$e(x) = e_i + \int_{x_i}^{x} e'(t)\,dt, \tag{30}$$

we have

$$|e(x)| \leqslant \|e_j\|+|x-x_i|\cdot\|e'\| \leqslant h\|y'\|+3|x-x_i|\cdot\|y'\| \leqslant (5/4)h\|y'\|$$

provided that $|x-x_i| \leqslant h/12$. Since we may always choose an i so that $|x-x_i| \leqslant h/12$ or $|x-\tau_i| \leqslant 5h/12$, the proof of (14) is complete.

To prove (15), we use (8) and the mean-value theorem. For each i, there is a u such that $\tau_i < u < \tau_{i+1}$ and

$$\begin{aligned} & c_i y_{i-1}'+3y_i'+a_i y_{i+1}'-8[y(\tau_{i+1})-y(\tau_i)]/(h_i+h_{i+1}) \\ &\quad = c_i y_{i-1}'+3y_i'+a_i y_{i+1}'-4y'(u) \\ &\quad = a_i[y_{i+1}'-y'(u)]+3[y_i'-y'(u)]-c_i[y'(u)-y_{i-1}']. \end{aligned}$$

Thus, from (8),

$$\begin{aligned} 2\|e_i'\| &\leqslant \|(a_i+c_i)\omega(y'; 3h/2)+3\omega(y'; h/2)\| \\ &= \omega(y'; 3h/2)+3\omega(y'; h/2) \leqslant 6\omega(y'; h/2). \end{aligned}$$

To prove (16), let Ly' denote the piecewise linear interpolant to y'. Then, for each x,

$$\begin{aligned} |e'(x)| &= |y'(x)-s'(x)| \leqslant |y'(x)-(Ly')(x)|+|(Ly')(x)-s'(x)| \\ &\leqslant |y'(x)-(Ly')(x)|+\|Ly'-s'\| = |y'(x)-(Ly')(x)|+\|(Ly')_i-s_i'\| \\ &= |y'(x)-(Ly')(x)|+\|y_i'-s_i'\| \leqslant |y'(x)-(Ly')(x)|+3\omega(y'; h/2) \end{aligned}$$

because of (15). Finally, Loginov [11] has shown that $\|g-Lg\| \leqslant (3/2)\omega(g; h/2)$ whenever Lg is the piecewise linear interpolant to g having corners (x_i, g_i) so that

$$\|e'\| \leqslant (3/2)\omega(y'; h/2)+3\omega(y'; h/2).$$

The proof of (17) is similar to the proof of (14). Either we have $|x-\tau_i| \leqslant (13/36)h$ and (29) so that

$$|e(x)| \leqslant |x-\tau_i|\cdot\|e'\| \leqslant (13/36)h\cdot(9/2)\omega(y'; h/2) = (13/8)h\omega(y'; h/2)$$

or we have $|x-x_i| \leqslant (5/36)h$ and (30) so that

$$\begin{aligned} |e(x)| &\leqslant \|e_j\|+|x-x_i|\cdot\|e'\| \\ &\leqslant h\omega(y'; h/2)+(5/36)h\cdot(9/2)\omega(y'; h/2) = (13/8)h\omega(y'; h/2). \end{aligned}$$

The proof of Theorem 2 is complete.

I.4. Proof of Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 2 except that Taylor's theorem with derivative remainder is used instead of the mean-value theorem.

We now prove (18). For a given i

$$\begin{aligned} & a_i y_{i-1}+3y_i+c_i y_{i+1}-4a_i y(\tau_i)-4c_i y(\tau_{i+1}) \\ &\quad = a_i(y_{i-1}-2y(\tau_i)+y_i)+2a_i(y_i-y(\tau_i))+2c_i(y_i-y(\tau_{i+1}))+ \\ &\qquad +c_i(y_i-2y(\tau_{i+1})+y_{i+1}) \\ &\quad = a_i h_i^2 y''(u_1)/4-a_i h_i^2 y''(u_2)/4-c_i h_{i+1}^2 y''(u_3)/4+c_i h_{i+1}^2 y''(u_4)/4 \end{aligned}$$

where $x_{i-1} < u_1 < x_i < u_4 < x_{i+1}$ and $\tau_i < u_2 < x_i < u_3 < \tau_{i+1}$. Thus, from (7),

$$2\|e_i\| \leqslant \|a_i h_i^2+c_i h_{i+1}^2\|\omega(y''; h)/4 = \|h_i h_{i+1}\|\omega(y''; h)/4 \leqslant (1/4)h^2\omega(y''; h).$$

To prove (19) we refer to (8) and note that

$$\begin{aligned} & c_i y_{i-1}'+3y_i'+a_i y_{i+1}'-8[y(\tau_{i+1})-y(\tau_i)]/(h_i+h_{i+1}) \\ &\quad = c_i[y_i'-h_i y''(u_1)]+3y_i'+a_i[y_i'+h_{i+1}y''(u_4)]- \\ &\qquad -8[y_i+h_{i+1}y_i'/2+h_{i+1}^2 y''(u_3)/8-y_i+h_i y_i'/2-h_i^2 y''(u_2)/8]/(h_i+h_{i+1}) \\ &\quad = c_i h_i[y''(u_2)-y''(u_1)]+a_i h_{i+1}[y''(u_4)-y''(u_3)] \end{aligned}$$

where $x_{i-1} < u_1 < x_i < u_4 < x_{i+1}$ and $\tau_i < u_2 < x_i < u_3 < \tau_{i+1}$. Thus,

$$2\|e_i'\| \leqslant \|c_i h_i+a_i h_{i+1}\|\omega(y''; h) \leqslant h\omega(y''; h).$$

To prove (20) we let Ly' be the piecewise linear interpolant to y'. For a given x we select i so that $|x-x_i| \leqslant h/2$. As in the proof of (16),

$$\begin{aligned}|e'(x)| &\leqslant |y'(x)-(Ly')(x)|+\|y_i'-s_i'\| \\ &= |y'(x)-(Ly')(x)|+\|e_i'\| \\ &= |x-x_i|\cdot|(y'-Ly')'(u_1)|+\|e_i'\| = |x-x_i|\cdot|y''(u_1)-y''(u_2)|+\|e_i'\| \\ &\leqslant (h/2)\omega(y'';h)+(h/2)\omega(y'';h)\end{aligned}$$

since $(y'-Ly')(x_i) = 0$ and $(Ly')'$ is piecewise constant.

The proof of (21) is similar to the proof of (14). With $|x-\tau_i| \leqslant (5/16)h$ and (29)

$$|e(x)| \leqslant |x-\tau_i|\cdot\|e'\| \leqslant (5/16)h^2\omega(y'';h).$$

With $|x-x_i| \leqslant (3/16)h$ and (30),

$$|e(x)| \leqslant \|e_j\|+|x-x_i|\cdot\|e'\| \leqslant (1/8)h^2\omega(y'';h)+(3/16)h^2\omega(y'';h).$$

The following proof of (22) comes from [9]. Since $s''(x)$ is constant on (x_{i-1}, x_i), u may be chosen in this interval so that

$$\begin{aligned}e''(x) &= y''(x)-y''(u)+y''(u)-s''(u) \\ &= y''(x)-y''(u)+e''(u) \\ &= y''(x)-y''(u)+[e_i'-e_{i-1}']/h_i.\end{aligned}$$

Thus,

$$|e''(x)| \leqslant \omega(y'';h)+2\|e_j'\|/h_i \leqslant \omega(y'';h)+(h/h_i)\omega(y'';h).$$

The proof of Theorem 3 is complete.

I.5. Proof of Theorem 4

Taylor's series gives

$$\begin{aligned}&a_i y_{i-1}+3y_i+c_i y_{i+1}-4a_i y(\tau_i)-4c_i y(\tau_{i+1}) \\ &\quad = a_i[y_i-h_i y_i'+h_i^2 y_i''/2-h_i^3 y'''(u_1)/6]+3y_i+ \\ &\qquad +c_i[y_i+h_{i+1}y_i'+h_{i+1}^2 y_i''/2+h_{i+1}^3 y'''(u_4)/6]- \\ &\qquad -4a_i[y_i-h_i y_i'/2+h_i^2 y_i''/8-h_i^3 y'''(u_2)/48]- \\ &\qquad -4c_i[y_i+h_{i+1}y_i'/2+h_{i+1}^2 y_i''/8+h_{i+1}^3 y'''(u_3)/48] \\ &\quad = a_i h_i^3[y'''(u_2)-2y'''(u_1)]/12+c_i h_{i+1}^3[2y'''(u_4)-y'''(u_3)]/12\end{aligned}$$

with $x_{i-1} < u_1 < x_i < u_4 < x_{i+1}$ and $\tau_i < u_2 < x_i < u_3 < \tau_{i+1}$. Thus, from (7)

$$2\|e_i\| \leqslant \|a_i h_i^3+c_i h_{i+1}^3\|\cdot\|y'''\|/4 \leqslant (1/4)h^3\|y'''\|$$

so that (23) is proved.

Similarly,

$$\begin{aligned}&c_i y_{i-1}'+3y_i'+a_i y_{i+1}'-8[y(\tau_{i+1})-y(\tau_i)]/(h_i+h_{i+1}) \\ &\quad = c_i[y_i'-h_i y_i''+h_i^2 y'''(u_1)/2]+3y_i'+ \\ &\qquad +a_i[y_i'+h_{i+1}y_i''+h_{i+1}^2 y'''(u_4)/2]- \\ &\qquad -8[y_i+h_{i+1}y_i'/2+h_{i+1}^2 y_i''/8+h_{i+1}^3 y'''(u_3)/48- \\ &\qquad -y_i+h_i y_i'/2-h_i^2 y_i''/8+h_i^3 y'''(u_2)/48]/(h_i+h_{i+1}) \\ &\quad = c_i h_i^2[3y'''(u_1)-y'''(u_2)]/6+a_i h_{i+1}^2[3y'''(u_4)-y'''(u_3)]/6\end{aligned}$$

so that (8) yields

$$2\|e_i'\| \leqslant \|c_i h_i^2+a_i h_{i+1}^2\|\cdot(2/3)\|y'''\| \leqslant (2/3)h^2\|y'''\|.$$

Thus, (24) is proved.

If Ly' is the piecewise linear interpolant to y' and if $x_{i-1} \leqslant x \leqslant x_i$, then $y'(x)-(Ly')(x) = (x-x_{i-1})(x_i-x)y'''(u)/2$ and, as in the proofs of (16) and (20),

$$|e'(x)| \leqslant |y'(x)-(Ly')(x)|+\|e_i'\| \leqslant (h^2/8)\|y'''\|+(1/3)h^2\|y'''\|$$

which is (25).

With $|x-\tau_i| \leqslant (17/44)h$ and (29),

$$|e(x)| \leqslant |x-\tau_i|\cdot\|e'\| \leqslant (17/44)h\cdot(11/24)h^2\|y'''\|.$$

With $|x-x_i| \leqslant (5/44)h$ and (30),

$$|e(x)| \leqslant \|e_j\|+|x-x_i|\cdot\|e'\| \leqslant (1/8)h^3\|y'''\|+(5/44)h\cdot(11/24)h^2\|y'''\|.$$

Thus, (26) is proved.

Concerning e'' we have, as in the proof of (22),

$$\begin{aligned}e''(x) &= y''(x)-y''(u_1)+(e_i'-e_{i-1}')/h_i \\ &= (x-u_1)y'''(u_2)+(e_i'-e_{i-1}')/h_i\end{aligned}$$

where the second equation comes from the mean-value theorem. Thus,

$$\begin{aligned}|e''(x)| &\leqslant |x-u_1|\cdot\|y'''\|+2\|e_j'\|/h_i \\ &\leqslant (h_i+(2/3)h^2/h_i)\|y'''\|\end{aligned}$$

and the proof of Theorem 4 is complete.

PART II

II.1. Exponential growth of quadratic null splines

This section is based on Birkhoff and de Boor [1], see also de Boor [3], but adapted to quadratic splines.

If $C(x)$ is a quadratic such that $C(h/2) = 0$, then

$$C(x) = C(0)(1-4x^2/h^2)+C'(0)(x-2x^2/h)$$

and

$$C'(x) = C(0)(-8x/h^2)+C'(0)(1-4x/h).$$

Thus,

$$\begin{bmatrix} C(h) \\ C'(h) \end{bmatrix} = -\begin{bmatrix} 3 & h \\ 8/h & 3 \end{bmatrix}\begin{bmatrix} C(0) \\ C'(0) \end{bmatrix}$$

and

$$\begin{bmatrix} C(0) \\ C'(0) \end{bmatrix} = \begin{bmatrix} -3 & h \\ 8/h & -3 \end{bmatrix} \begin{bmatrix} C(h) \\ C'(h) \end{bmatrix}.$$

A paraphrase of this reasoning yields:

LEMMA 1. *Let C be a quadratic spline. If $C(\tau_j) = 0$, then*

$$\begin{bmatrix} C_j \\ C_j' \end{bmatrix} = -\begin{bmatrix} 3 & h_j \\ 8/h_j & 3 \end{bmatrix} \begin{bmatrix} C_{j-1} \\ C_{j-1}' \end{bmatrix}$$

and

$$\begin{bmatrix} C_{j-1} \\ C_{j-1}' \end{bmatrix} = \begin{bmatrix} -3 & h_j \\ 8/h_j & -3 \end{bmatrix} \begin{bmatrix} C_j \\ C_j' \end{bmatrix}.$$

As in Part I, $C_j = C(x_j)$, $h_j = x_j - x_{j-1}$, etc.

If C is a nonperiodic quadratic spline with a doubly-infinite set of knots such that $C(\tau_j) = 0$ for all j, then we shall say that C is a *quadratic null spline*. Quadratic null splines have an exponential growth property in the following sense:

LEMMA 2. *Let C be a quadratic null spline. If i is such that $C_i C_i' \geq 0$, then*

$$\begin{bmatrix} |C_j| \\ |C_j'| \end{bmatrix} \geq 3^{j-i} \begin{bmatrix} |C_i| \\ |C_i'| \end{bmatrix} \quad \text{for all } j > i.$$

If i is such that $C_i C_i' \leq 0$, then

$$\begin{bmatrix} |C_j| \\ |C_j'| \end{bmatrix} \geq 3^{i-j} \begin{bmatrix} |C_i| \\ |C_i'| \end{bmatrix} \quad \text{for all } j < i.$$

Moreover, for all i, $\max\{|C(x)|: x_{i-1} \leq x \leq x_i\} = \max\{|C_{i-1}|, |C_i|\}$.

Proof. If $C_i C'_i \geq 0$, then the first matrix equation in Lemma 1 gives

$$\begin{bmatrix} |C_{i+1}| \\ |C_{i+1}'| \end{bmatrix} = \begin{bmatrix} |3C_i + h_{i+1} C_i'| \\ |8C_i/h_{i+1} + 3C_i'| \end{bmatrix} = \begin{bmatrix} 3|C_i| + h_{i+1}|C_i'| \\ (8/h_{i+1})|C_i| + 3|C_i'| \end{bmatrix} \geq \begin{bmatrix} 3|C_i| \\ 3|C_i'| \end{bmatrix} = 3\begin{bmatrix} |C_i| \\ |C_i'| \end{bmatrix}.$$

In addition, $C_{i+1} C_{i+1}' \geq 0$. Thus, the process may be repeated so that the first matrix relation of Lemma 2 follows by mathematical induction.

The second matrix relation follows in like manner. The last assertion of Lemma 2 follows from the geometry of parabolas.

II.2. The fundamental splines of interpolation

The set of functions $\{C^i: i = 1, \ldots, n\}$ defined by

(i) C^i is a periodic quadratic spline having knots in Δ;

(ii) $C^i(\tau_j) = 0$ if $j \neq i$; and

(iii) $C^i(\tau_i) = 1$

are said to be *fundamental quadratic splines*. If $y \in L_\infty[0, 1]$ and $s = P_\Delta y$ is its periodic quadratic spline interpolant, then

$$s = \sum_{i=1}^{n} y(\tau_i) C^i.$$

It is well-known that $||P_\Delta|| = \max\{\sum |C^i(x)|: 0 \leq x \leq 1\}$.

In Theorem 1 it was shown that $||P_\Delta|| \leq 2$. We shall use quadratic null splines to prove the weaker assertion $||P_\Delta|| \leq 14$. Our purpose in doing this is that it may be easier to extend the null spline argument than to extend the argument in the proof of Theorem 1.

We shall also use null splines to reprove one of the local error bound theorems of [9].

II.3. Operator norm bounds via null splines

For $i = 2, 3, \ldots, n$ the fundamental spline C^i coincides with a quadratic null spline on $[0, x_{i-1}]$. For $i = 1, 2, \ldots, n-1$, C^i coincides with another null spline on $[x_i, 1]$. By extending the knot sequence Δ periodically, we may combine these facts to state that C^i coincides with a quadratic null spline on $[x_i, x_{n+i-1}]$. Then, Lemma 2 applies on this interval so that we may compare $C_k^i = C^i(x_k)$ with $C_i^i = C^i(x_i)$ or $C_{i-1}^i = C^i(x_{i-1})$. The result is

LEMMA 3. *If $i \leq k \leq n$, then either $|C_k^i| \leq 3^{k+1-i-n}|C_{i-1}^i|$ or $|C_k^i| \leq 3^{i-k}|C_i^i|$. If $0 \leq k \leq i-1$, then either $|C_k^i| \leq 3^{k+1-i}|C_{i-1}^i|$ or $|C_k^i| \leq 3^{i-k-n}|C_i^i|$.*

Similar relations hold for the derivative of C^i, but we shall not need them.

Now, $||P_\Delta|| = \sum |C^i(x^*)|$ for some x^* on $[0, 1]$. There is a k such that $x_{k-1} \leq x^* \leq x_k$. Then

$$\begin{aligned}
||P_\Delta|| &\leq |C^k(x^*)| + \sum_{i \neq k} \max\{|C_{k-1}^i|, |C_k^i|\} \\
&\leq |C^k(x^*)| + \sum_{1}^{k-1} \max\{3^{k+1-i-n}|C_{i-1}^i|, 3^{i+1-k}|C_i^i|\} + \\
&\quad + \sum_{k+1}^{n} \max\{3^{k+1-i}|C_{i-1}^i|, 3^{i+1-k-n}|C_i^i|\} \\
&\leq |C^k(x^*)| + \sum_{1}^{k-1} (3^{k+1-i-n} + 3^{i+1-k}) \max\{||C_{j-1}^j||, ||C_j^j||\} + \\
&\quad + \sum_{k+1}^{n} (3^{k+1-i} + 3^{i+1-k-n}) \max\{||C_{j-1}^j||, ||C_j^j||\} \\
&\leq |C^k(x^*)| + [(3/2) + (3/2) + (3/2) + (3/2)] \max\{||C_{j-1}^j||, ||C_j^j||\} \\
&\leq 2 + 6 \cdot 2 = 14,
\end{aligned}$$

where the last line follows from a paraphrase of the proof of (9) in Theorem 1.

A more careful argument, using (2) and the properties of the fundamental splines, will produce a smaller number than 14, but I see no way to produce the "best" constant 2 of Theorem 1 above by using a null spline argument.

II.4. Local error bounds via null splines

In Kammerer, Reddien, and Varga [9], a matrix result of Kershaw [10] is exploited to establish local rates of convergence for quadratic spline interpolation to a bounded function y which is only locally smooth. We shall prove a theorem similar to [9], Th. 5.1, using null splines. Using cubic null splines, de Boor [3] has proved similar results for cubic spline interpolation.

We note here that [9], Relation (5.2) seems to be incorrectly stated since it is not sufficient to deduce the lines which follow [9], Relation (5.12).

Let $I^* = [a^*, b^*]$ where $0 \leqslant a^* < b^* < 1$. Let a partition Δ of $[0, 1]$ be given. Let y and z be bounded functions satisfying

(i) $y \in C^k[a^*, b^*]$ for some fixed $k = 0, 1, 2$, or 3;

(ii) $z = y$ on I^*;

(iii) $z \in C^k[0, 1]$; and

(iv) $\omega(z^{(k)}; \delta) = \omega^*(y^{(k)}; \delta)$ if $k < 3$ or $\|z^{(k)}\|_{L_\infty[0,1]} = \|y^{(k)}\|_{L_\infty[a^*, b^*]}$ if $k = 3$.

Here, $\omega^*(g^{(k)}; \delta)$ is the modulus of continuity of $g^{(k)}$, the k-derivative of g, on the interval I^*.

Following [9], we replace the periodicity condition (c) in the definition of quadratic spline interpolation with the interpolatory condition

(c)′ $s(0) = y(0)$ and $s(1) = y(1)$

and let $s = P_\Delta y$ be the corresponding quadratic spline interpolant to y.

Let p and q be integers such that $\tau_p < a^* \leqslant \tau_{p+1}$ and $\tau_{q-1} \leqslant b^* < \tau_q$. For a given x in I^*, let $r = r(x)$ be an integer such that $x_{r-1} \leqslant x \leqslant x_r$.

THEOREM 5. *Let y and z satisfy* (i)–(iv). *Let $x \in I^*$ and let p, q, r be as above. Let $s = P_\Delta y$ be the quadratic spline interpolant to y based on* (c)′. *Then*

$$|(y-P_\Delta y)(x)| \leqslant 3h^k\omega^*(y^{(k)}; h/2)+3\|z-y\|\cdot(3^{p-r}+3^{r-q})$$

if $k = 0, 1$, or 2; *and*

$$|y-P_\Delta y)(x)| \leqslant (17/96)h^3\|y'''\|_{L_\infty[a^*, b^*]}+3\|z-y\|\cdot(3^{p-r}+3^{r-q})$$

if $k = 3$.

Proof. Let $\sum'$ denote summation over $\{i: \tau_i \notin I^*\}$. Using the fundamental splines above, as modified for (c)′, we have

$$\begin{aligned}|e(x)| &= \Big|y(x)-\sum_{i=0}^{n+1} y(\tau_i)C^i(x)\Big| \\ &= \Big|z(x)-\sum z(\tau_i)C^i(x)+\sum{}' [z(\tau_i)-y(\tau_i)]C^i(x)\Big| \\ &\leqslant \|z-P_\Delta z\|+\|z-y\|\sum{}' |C^i(x)| \\ &\leqslant \|z-P_\Delta z\|+\|z-y\|\cdot\max\{\|C^j_{j-1}\|, \|C^j_j\|\}\cdot\Big(\sum_{i=0}^{p-1} 3^{i+1-r}+\sum_{q+1}^{n+1} 3^{r+1-i}\Big).\end{aligned}$$

The last inequality comes from Lemma 3 of the preceding section as modified to correspond to (c)′. From (9) in Theorem 1 above

$$\begin{aligned}|e(x)| &\leqslant \|z-P_\Delta z\|+\|z-y\|\cdot 2\cdot(3/2)\,(3^{p-r}+3^{r-q}) \\ &= \|z-P_\Delta z\|+3\|z-y\|\cdot(3^{p-r}+3^{r-q}).\end{aligned}$$

Now, the versions of (11), (17), (21), and (26) above corresponding to (c)′ give bounds on $\|z-P_\Delta z\|$. As noted in the introduction, the constants are the same. With (iv) we have the result stated in the theorem.

With a more careful argument, the factor $3\|z-y\|$ may be exchanged for $2\|z-y\|$. We will not give this argument.

PART III

An important set of periodic quadratic splines are the periodic B-splines $\{N^j: j = 1, \ldots, n\}$. These are the same B-splines which appear throughout the spline literature, extended periodically. They have the "partition-of-unity" property:

$$\sum |N^j(x)| = \sum N^j(x) = 1 \quad \text{for} \quad 0 \leqslant x \leqslant 1.$$

For our purposes, it is convenient to define them as

$$\begin{aligned}N^i = (1/4)C^ih_i/(h_i+h_{i+1})+(1/4)C^{i+1}[4-h_{i+1}/(h_i+h_{i+1})-h_{i+1}/(h_{i+1}+h_{i+2})]+ \\ +(1/4)C^{i+2}h_{i+2}/(h_{i+1}+h_{i+2}),\end{aligned}$$

for $i = 1, 2, \ldots, n$. Of course, the C^j are the periodic fundamental quadratic splines and we have identified $C^1 = C^{n+1}$, $C^2 = C^{n+2}$, $h_1 = h_{n+1}$, and $h_2 = h_{n+2}$.

We shall use the B-splines to provide another proof that $\|P_\Delta\|$ is bounded. See also de Boor [2].

Let $A = [a_{ij}]$ be the $n\times n$ matrix defined by $a_{ij} = N^i(\tau_{j+1})$. Let $B = [b_{ij}] = A^{-1}$. Now,

$$\begin{aligned}\|P_\Delta\| &= \sum_i |C^i(x^*)| = \sum_i\Big|\sum_j b_{ij}N^j(x^*)\Big| \leqslant \sum_j\Big(\sum_i |b_{ij}|\Big)N^j(x^*) \\ &\leqslant \max_j \sum_i |b_{ij}| = \|B\|_1 \quad \text{(by definition of matrix norm)} \\ &\leqslant \|D\|_1/(1-\|I-DA\|_1)\end{aligned}$$

provided that D is chosen so that $\|I-DA\|_1 < 1$.

The choice $D = \operatorname{diag}\{1/a_{ii}\}$ gives $\|D\|_1 = 1/\min\{a_{ii}\} \leqslant 2$ and

$$\|I-DA\|_1 = \max\{a_{i,i+1}/a_{ii}+a_{i+2,i+1}/a_{i+2,i+2}\}.$$

Without loss of generality, we may assume this maximum is attained for $i = 1$

so that

$$\begin{aligned}
\|I-DA\|_1 &= a_{12}/a_{11}+a_{32}/a_{33}\\
&= [h_3/(h_2+h_3)]/[4-h_2/(h_1+h_2)-h_2/(h_2+h_3)]+\\
&\quad +[h_3/(h_3+h_4)]/[4-h_4/(h_3+h_4)-h_4/(h_4+h_5)]\\
&< [h_3/(h_2+h_3)]/[3-h_2/(h_2+h_3)]+\\
&\quad +[h_3/(h_3+h_4)]/[3-h_4/(h_3+h_4)]\\
&= h_3/(2h_2+3h_3)+h_3/(3h_3+2h_4)\\
&< 2/3.
\end{aligned}$$

Thus, we have shown that $\|P_\Delta\| \leqslant 2/(1-2/3) = 6$.

This result should be contrasted with Theorem 1 above.

As we observed in the Introduction, de Boor [2] has exploited total positivity to prove a similar cubic spline result.

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Presented to the Semester
Approximation Theory
September 17–December 17, 1975