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QUANTITATIVE ESTIMATES OF N. N. LUZIN'S C-PROPERTY FOR CLASSES OF INTEGRABLE FUNCTIONS

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1. Introduction

N. N. Luzin's theorem (see [2], and also [4]) asserts that an arbitrary measurable function which is finite almost everywhere, may be considered as a continuous function, if some set of points is ignored, the Lebesgue measure of which may be made arbitrarily small. More exactly, for an arbitrary function $f(x)$, measurable and finite a.e. on $[0, 1]$, and for an arbitrary number $\varepsilon > 0$, there exist a perfect set and a continuous function $g(x) = g_{\varepsilon,f}(x)$ $(x \in [0, 1])$ such that

$$\text{meas}\, F \geqslant 1-\varepsilon \quad \text{and} \quad g(x) = f(x) \quad \text{for } x \in F. \tag{1}$$

Although it is far from the contents of this paper, a very deep result directed to the further investigations of properties possessed by functions g from Luzin's C-property (1), may be noticed here. D. E. Menshov [3] proved (in the case of periodic functions f) that the set F and the function g in (1) may be chosen in such a way that the trigonometric Fourier series of g converges uniformly.

The aim of this paper is to get estimates of the uniform modulus of continuity of function g in (1) under the assumption that the original function f is not only measurable and finite a.e. on $[0, 1]$ (as required by Luzin's theorem), but does also belong to the space L^p $[0, 1]$, for some $p \geqslant 1$. These "quantitative estimates of C-property" will then be expressed by means of L^p-modulus of continuity of the function f.

Some particular consequences from the observations made below, sound as follows.

Let $1 \leqslant p < \infty$, *and assume that* $f(x) \in \text{Lip}(\alpha, p)$*, *where* $0 < \alpha < 1$. *Then, for each* $\varepsilon > 0$ *and each* $\eta > 0$, *there exist a set* $F = F_{\varepsilon,\eta,f}$ *with* $\text{meas}\, F > 1-\varepsilon$ *and a continuous function* $g(x) = g_{\varepsilon,\eta,f}(x)$ *such that*

$$g(x) = f(x) \quad (x \in F)$$

* More careful definitions will be found in § 2 below.

and, for the uniform modulus of continuity of g

$$\omega_\infty(g,\delta) = \sup\{|g(x)-g(y)|:\ |x-y|\leqslant\delta\},$$

the following estimate holds

$$\omega_\infty(g,\delta) = o\big(\delta^\alpha(\log(1/\delta))^{(1+\eta)/p}\big) \quad (\delta\to 0). \tag{2}$$

On the other hand, it will be shown that *for no values of* p, $1\leqslant p<\infty$, *and* α, $0<\alpha<1$, *the value* $\eta=0$ *in* (2) *is permitted.* In particular, logarithmic factors which tend to $+\infty$ as $\delta\to 0$, do exist, by necessity, at δ^α in the right-hand side of (2), at least for some functions from the classes $\mathrm{Lip}(\alpha,p)$.

Furthermore, in the limiting case $\alpha=1$ a more precise estimate of $\omega_\infty(g,\delta)$ than (2), may be easily obtained:

$$\omega_\infty(g,\delta) = O(\delta) \quad (\delta\to 0), \tag{3}$$

the factor in O depending on ε. This estimate is a consequence of the embedding theorem due to G. H. Hardy and J. E. Littewood [1], (see also [7]), according to which the condition $f\in\mathrm{Lip}(1,1)$ implies that $f(x)$ coincides, for almost all x, with some function $f_1(x)$ of bounded variation over $[0,1]$ and, in particular, with f_1, differentiable a.e.

A comparison of this remark with (2) shows that the scale of $\mathrm{Lip}(\alpha,p)$-classes for $0<\alpha<1$ is too rough to provide a smooth pastening to the case of $\mathrm{Lip}(1,p)$-class, and that some more accurate classification of moduli of continuity is needed. For purposes of the latter, a function $\Omega(h)$ is introduced in § 2, which controlls the contemporary behaviour of $\omega_p(f,\delta)$ and $\omega_p(f,\delta)/\delta$.

In § 3 the proof of the estimate from above (Theorem 1 of § 2) is carried out. This proof uses approximation properties of V. A. Steklov averages

$$S_\delta(f,x) = \frac{1}{\delta}\int\limits_{-\delta/2}^{\delta/2} f(x+t)\,dt \tag{4}$$

in L_p-norm, and also Hardy–Littlewood maximal theorems (see [7], p. 32).

In § 4 Theorem 2 (see § 2) is proved, which shows the sharpness of Theorem 1.

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2. Notation and results

Without any loss of generality we shall assume in what follows that all the functions are periodic with period 1. Furthermore, let $p\geqslant 1$ and let $\|f\|_p$ denote the L_p-norm of a function f from L_p, i.e.,

$$\|f\|_p = \Big(\int\limits_0^1 |f(x)|^p\,dx\Big)^{1/p}$$

and denote by $\omega_p(f,\delta)$ the L_p-modulus of continuity of $f(x)$:

$$\omega_p(f,\delta) = \sup\{\|f(x)-f(x+h)\|_p:\ 0<h\leqslant\delta\}.$$

Denote by $H_{p,\omega}$ the class of functions f, defined as follows:

$$H_{p,\omega} = \{f\in L_p:\ \omega_p(f,\delta)\leqslant\omega(\delta),\ 0<\delta\leqslant\tfrac12\},$$

where $\omega(\delta)$ is an arbitrary function of "continuous modulus of continuity" type (see [5]). In particular, if $\omega(\delta)=\delta^\alpha$, where $0<\alpha\leqslant 1$, we get the $\mathrm{Lip}(\alpha,p)$-class.

As it was already mentioned above, we are going to obtain sharp estimates for $\omega_\infty(g,\delta)$ in the classes $H_{p,\omega}$, not equivalent to the class $\mathrm{Lip}(1,p)$, that is, in the classes defined by moduli of continuity $\omega(\delta)$, for which the following condition holds:

$$\frac{\omega(\delta)}{\delta}\to+\infty \quad (\delta\to 0). \tag{5}$$

To fulfil it, a definition of the function $\Omega(\delta)$ controlling $\omega(\delta)$ and $\omega(\delta)/\delta$ is needed, which is done in (8) and (11) below. First of all choose, using (5), some sequence $\{\delta_k\}_{k=0}^\infty$ of positive numbers $\delta_{k'}$, $\delta_k\downarrow 0$ $(k\to\infty)$, with the property

$$\frac{1}{c}\omega(\delta)\leqslant\sum_{k=0}^\infty\min\Big(\frac{\omega(\delta_k)}{\delta_k}\delta,\ \omega(\delta_k)\Big)\leqslant c\,\omega(\delta) \quad (0\leqslant\delta\leqslant\tfrac12) \tag{6}$$

where c is an absolute positive constant >1. Formally, the definition of $\{\delta_k\}_{k=0}^\infty$ is made as follows: let

$$\omega^*(\delta) = \frac{\delta}{\omega(\delta)} \quad (0<\delta\leqslant\tfrac12) \tag{7}$$

and let

$$\delta_0=\frac12,\qquad \delta_{k+1}=\min\Big\{\delta:\ \max\Big(\frac{\omega(\delta)}{\omega(\delta_k)},\ \frac{\omega^*(\delta)}{\omega^*(\delta_k)}\Big)=\frac14\Big\} \quad (k=0,1,\ldots). \tag{8}$$

Using (5), according to which $\omega^*(\delta)\to 0$ $(\delta\to 0)$, and the inequality

$$\omega^*(\delta)\leqslant 2\omega^*(\eta) \quad (0<\delta\leqslant\eta\leqslant\tfrac12) \tag{9}$$

which is a consequence from subadditivity of moduli of continuity, we infer that the sequence $\{\delta_k\}$ tends to 0 monotonuously as $k\to\infty$, and moreover, that

$$\delta_{k+1}\leqslant\tfrac12\delta_k \quad (k=0,1,\ldots). \tag{10}$$

Define

$$\Omega(\delta) = 2\cdot 4^{-k} \quad (\delta_{k+1}<\delta\leqslant\delta_k,\ k=0,1,\ldots);\qquad \Omega(0)=0 \tag{11}$$

(see also [6]). It may easily be checked that

$$\Omega(\delta)\geqslant\max\Big\{\frac{\omega(\delta)}{\omega(\frac12)},\ \frac{\delta\,\omega(\frac12)}{\omega(\delta)}\Big\} \quad (0\leqslant\delta\leqslant\tfrac12) \tag{12}$$

and that (6) holds, with $c=8$.

THEOREM 1. *Let $p \geqslant 1$, and let $\omega(\delta) = \omega_p(f, \delta)$ be the L_p-modulus of continuity of a function $f \in L_p$. Suppose that*

$$\frac{\omega(\delta)}{\delta} \to +\infty \qquad (\delta \to 0)$$

and that $\Omega(\delta)$ is defined by (8) *and* (11). *Then, with an arbitrary function $\psi(u)$, non-increasing and positive for $u \geqslant 0$, which satisfies the condition*

$$\int_0^1 \frac{du}{u\psi(u)} < \infty, \tag{13}$$

the following estimate holds

$$|f(x)-f(y)| \leqslant \big(C(x)+C(y)\big)\omega(|x-y|)\psi^{1/p}\big(\Omega(|x-y|)\big), \tag{14}$$

where the function $C(x) = C_{f,\psi}(x)$ is nonnegative, finite a.e. *and upper semicontinuous. Moreover, if $p > 1$, then $C(x) \in L_p$, and if $p = 1$, then*

$$\operatorname{meas}\{x\colon x \in [0, 1],\, C(x) > z\} \leqslant c/z \qquad (z > 0),$$

where c is a positive constant, $c = c_\psi$.

COROLLARY 1. *For each $\varepsilon > 0$ and each $\eta > 0$ there exist a set $F = F_{\varepsilon,\eta,f}$ and a continuous function $g(x) = g_{\varepsilon,\eta,f}(x)$ such that $F \cap [0, 1]$ is closed,*

$$g(x) = f(x) \quad (x \in F), \qquad \operatorname{meas} F \cap [0, 1] \geqslant 1-\varepsilon, \tag{15}$$

and that the uniform modulus of continuity of g satisfies the condition

$$\omega_\infty(g, \delta) = O\left[\omega(\delta)\log^{(1+\eta)/p}\min\left(\frac{\omega(\frac{1}{2})}{\omega(\delta)}, \frac{\omega(\delta)}{\delta\omega(\frac{1}{2})}\right)\right], \tag{16}$$

$$(\delta \to 0;\ \omega(\delta) = \omega_p(f, \delta)).$$

Proof. To deduce this statement from Theorem 1, let $\psi(u) = \big(\log(e/u)\big)^{1+\eta/2}$ $(0 < u \leqslant 1)$. Then, by (11),

$$\omega(\delta)\psi^{1/p}\big(\Omega(\delta)\big) = \omega(\delta)(c_1+c_2k)^{(1+\eta/2)(1/p)}$$
$$(\delta_{k+1} < \delta \leqslant \delta_k,\ k = 0, 1, \ldots;\ c_1 = \log(e/2),\ c_2 = \log 4)$$

and thus, taking (8) into account, we see that

$$\omega(\gamma)\psi^{1/p}\big(\Omega(\gamma)\big) \leqslant c_\eta\omega(\delta)\psi^{1/p}\big(\Omega(\delta)\big) \qquad (0 < \gamma \leqslant \delta \leqslant \tfrac{1}{2}) \tag{17}$$

with c_η independent from δ.

Notice that (13) holds for the function ψ. Let $z > 0$ and let G_z denote the set of points x, where $C(x) > z$, $C(x)$ being the factor from the right-hand side of (14). By Theorem 1, G_z is an open set, $\operatorname{meas} G_z \cap [0, 1] \to 0$ $(z \to \infty)$ and thus, for some $z = z(\varepsilon)$ large enough,

$$\operatorname{meas} G_z \cap [0, 1] \leqslant \varepsilon.$$

Let F be the complement of G_z, and let (a_i, b_i) be the sequence of consistuent intervals of G_z. Define $g(x)$ as follows:

$$g(x) = \begin{cases} f(x) \quad (x \in F), \\ \text{linear function in each } (a_i, b_i) \text{ with } g(a_i) \\ \qquad = f(a_i),\ g(b_i) = f(b_i). \end{cases} \tag{18}$$

It is obvious that (15) holds for $g(x)$, and thus we see from (12) that, to prove (16), it would suffice to verify that

$$\omega_\infty(g, \delta) \leqslant c'_\eta z\omega(\delta)\psi^{1/p}\big(\Omega(\delta)\big) \qquad (0 \leqslant \delta \leqslant \tfrac{1}{2}) \tag{19}$$

with some c'_η independent from δ. Define

$$\xi_1(x) = \min\{\xi\colon \xi \in F,\ \xi \geqslant x\},$$
$$\xi_2(x) = \max\{\xi\colon \xi \in F,\ \xi \leqslant x\}.$$

Then,

$$|g(x)-g(y)| \leqslant |g(x)-g(\xi_1(x))|+|g(\xi_1(x))|-|g(\xi_2(y))|+|g(\xi_2(y))-g(y)|. \tag{20}$$

Suppose that $0 \leqslant y-x \leqslant \delta$. There are 4 possibilities for x and y:

(1) $x \in F,\ y \in F$; (2) $x \in F,\ y \notin F$; (3) $x \notin F,\ y \in F$; (4) $x \notin F,\ y \notin F$.

In case (1) we get the estimate

$$|g(x)-g(y)| \leqslant c'_\eta z\omega(\delta)\psi^{1/p}\big(\Omega(\delta)\big), \tag{21}$$

with $c'_\eta = 2c_\eta$, c_η being the same as in (17), directly from (18), (17) and the definition of F. Now consider case (4) only, since the remaining two cases may be treated quite analogously. By (18) and (14),

$$\begin{aligned} |g(x)-g(\xi_1(x))| &= \frac{\xi_1(x)-x}{\xi_1(x)-\xi_2(x)}\,|g(\xi_2(x))-g(\xi_1(x))| \\ &= \frac{\xi_1(x)-x}{\xi_1(x)-\xi_2(x)}\,|f(\xi_2(x))-f(\xi_1(x))| \\ &\leqslant 2z\,\frac{\xi_1(x)-x}{\xi_1(x)-\xi_2(x)}\,\omega\big(\xi_1(x)-\xi_2(x)\big)\psi^{1/p}\Big(\Omega\big(\xi_1(x)-\xi_2(x)\big)\Big), \end{aligned} \tag{22}$$

and since $\xi_2(x) < x < \xi_1(x) \leqslant \xi_2(y) < y < \xi_1(y)$, we see from (22), (9), and (17) that the first term on the right-hand in (20) does not exceed

$$4c_\eta\omega(\delta)\psi^{1/p}\big(\Omega(\delta)\big).$$

By the same reason, this estimate is also valid for the last term on the right-hand in (20), while for the second term (21) may be already applied. Thus, by addition, (21) follows with $c'_\eta = 10c_\eta$. This completes the proof of (19), and of Corollary 1, too.

THEOREM 2. *Suppose that $p \geqslant 1$, and let $\omega(\delta)$ be a modulus of continuity with*

$$\frac{\omega(\delta)}{\delta} \to +\infty \qquad (\delta \to 0). \tag{23}$$

Let $\Omega(\delta)$ be defined by (8) *and* (11), *and suppose that for a function $\psi(u)$, which is positive and non-increasing for $u \geqslant 0$, the following condition holds*

$$\int_0^1 \frac{du}{u\psi(u)} = +\infty. \tag{24}$$

Then there exists a function f in the class $H_{p,\omega}$ such that for an arbitrary measurable set F the relation

$$\overline{\lim_{\substack{x_1\to x,\, x_2\to x\\ x_1, x_2\in F}}} \frac{|f(x_1)-f(x_2)|}{\omega(|x_1-x_2|)\psi^{1/p}(\Omega(|x_1-x_2|))} = +\infty \tag{25}$$

holds at arbitrary point x which is a point of density for the set F.

COROLLARY 2. *If $1 \leqslant p < \infty$ and if a modulus of continuity $\omega(\delta)$ satisfies* (23), *then there exists a function $f(x)$ in the class $H_{p,\omega}$ such that, if some continuous function $g(x)$ coincides with $f(x)$ on some set of positive measure, then*

$$\overline{\lim_{\delta\to 0}} \frac{\omega_\infty(g, \delta)}{\omega_p(f, \delta)} = +\infty.$$

COROLLARY 3. *If $1 \leqslant p < \infty$ and $0 < \alpha < 1$, then there exists a function $f(x)$ in the class* Lip(α, p) *such that, if some continuous function $g(x)$ coincides with $f(x)$ on some set of positive measure, then*

$$\lim_{\delta\to 0} \frac{\omega_\infty(g, \delta)}{\delta^\alpha \log^{1/p}(1/\delta)} = +\infty.$$

Corollary 1 shows that, in general, for no class $H_{p,\omega}$ $(1 \leqslant p < \infty)$, except for the class Lip$(1, p)$, the estimate

$$\omega_\infty(g, \delta) = O(\omega_p(f, \delta))$$

is valid for the uniform modulus of continuity of the function g, which provides Luzin's C-property.

Corollaries 1 and 3 imply in particular the statements formulated in the introduction for the classes Lip(α, p).

3. Proof of Theorem 1

Let $M(g, x)$ stand for the Hardy–Littlewood maximal function of some $g \in L[0, 1]$, that is,

$$M(g, x) = \sup_{I\ni x} \frac{1}{\operatorname{meas} I} \int_I |g(t)|\,dt,$$

the sup being taken over all the intervals, containing x (see [7]).

Theorem 1 will be deduced from the following statement.

LEMMA. *Suppose that* (5) *holds for $\omega(\delta) = \omega_p(f, \delta)$ and that the sequence $\{\delta_k\}_{k=0}^\infty$ is defined by* (8). *There exists a sequence of absolutely continuous functions $\{f_k(x)\}_{k=0}^\infty$ such that for an arbitrary numerical sequence $\{a_k\}_{k=0}^\infty$, $a_k \geqslant 0$, for which*

$$\sum_{k=0}^\infty a_k^p < \infty, \tag{28}$$

both the functions

$$B_1(x) = B_1(a, f, x) = \sup_{k\geqslant 0} \frac{M(f-f_k, x)}{\omega(\delta_{k+1})} a_k$$

and

$$B_2(x) = B_2(a, f, x) = \sup_{k\geqslant 0} \frac{M(f_k', x)\delta_k}{\omega(\delta_k)} a_k$$

are finite a.e. *Moreover, if $p > 1$, then $B_1, B_2 \in L_p$, and if $p = 1$, then the weak type estimate holds for B_1 and B_2:*

$$\sup\{z \operatorname{meas}\{x\colon\ x\in[0,1],\, B_1(x) > z\}\colon\ z > 0\} \leqslant c\sum_{k=0}^\infty a_k,$$

$$\sup\{z \operatorname{meas}\{x\colon\ x\in[0,1],\, B_2(x) > z\}\colon\ z > 0\} \leqslant c\sum_{k=0}^\infty a_k.$$

To prove this statement, notice that by (8) at least one of the following two possibilities occurs for each $k \geqslant 0$:

$$(1)\ \omega(\delta_{k+1}) = \frac{1}{4}\omega(\delta_k) \quad\text{or}\quad (2)\ \frac{\delta_{k+1}}{\omega(\delta_{k+1})} = \frac{1}{4}\frac{\delta_k}{\omega(\delta_k)}. \tag{29}$$

If for some $k \geqslant 0$ we have (1), then let

$$f_k(x) = S_{\delta_k}(f, x)$$

where $S_\delta(f, x)$ is Steklov average (4). For all extra values of k, let

$$f_k(x) = S_{\delta_{k+1}}(f, x).$$

If we apply the following well-known L_p-estimates:

$$\|f - S_\delta(f)\|_p \leqslant \omega_p(f, \delta), \quad \|S_\delta'(f)\| \leqslant \frac{\omega_p(f, \delta)}{\delta} \tag{30}$$

(see [7], p. 117), we see from (29) that

$$\|f - f_k\|_p \leqslant 4\omega(\delta_{k+1});\ \|f_k'\|_p \leqslant 4\frac{\omega(\delta_k)}{\delta_k} \quad (k = 0, 1, \ldots). \tag{31}$$

To get the assertions of the Lemma, it is sufficient now to apply the Hardy–Litllewood maximal theorems (see [7], p. 32) according to which

$$\left\| M\frac{g}{\|g\|_p} \right\| \leqslant c_p \quad (p > 1) \tag{32}$$

and

$$\sup_{z>0}\left\{z\,\mathrm{meas}\left\{x\colon\ x\in[0,1],\ M\left(\frac{g}{\|g\|_1},x\right)>z\right\}\colon\ z>0\right\}\leqslant c, \tag{33}$$

where c_p depends on p only, and c is an absolute positive constant. If $p>1$, we write

$$\int_0^1 B_1^p(x)\,dx\leqslant\sum_{k=0}^{\infty}a_k^p\int_0^1\frac{M^p(f-f_k,x)}{\omega^p(\delta_{k+1})}\,dx,$$

and thus, by (31) and (32),

$$\|B_1\|_p\leqslant 4c_p\Big(\sum_{k=0}^{\infty}a_k^p\Big)^{1/p}. \tag{34}$$

If $p=1$, then an application of (33), (31) and the trivial inequality

$$\mathrm{meas}\,\{x\colon\ x\in[0,1],\ B_1(x)>z\}\leqslant\sum_{k=0}^{\infty}\mathrm{meas}\left\{x\colon\ x\in[0,1],\ \frac{M(f-f_k,x)}{\omega(\delta_{k+1})}a_k>z\right\}$$

shows the validity of the Lemma for B_1. Estimates for B_2 are obtained in a quite analogous manner.

Now we complete the proof of Theorem 1. Let

$$a_k=\frac{1}{\psi^{1/p}(2\cdot 4^{-k})}\qquad (k=0,1,\ldots).$$

It follows from (13) that (28) holds for a_k's chosen. Furthermore, let x, y be fixed and suppose that

$$\delta_{k+1}<|x-y|\leqslant\delta_k. \tag{35}$$

Write

$$|f(x)-f(y)|\leqslant|f(x)-f_k(x)|+|f_k(x)-f_k(y)|+|f_k(y)-f(y)|,$$

and apply Lemma to estimate the right-hand side. We get

$$|f(x)-f(y)|\leqslant\left[\big(B_1(x)+B_1(y)\big)\omega(\delta_{k+1})+B_2(x)|x-y|\frac{\omega(\delta_k)}{\delta_k}\right]\psi^{1/p}(2\cdot 4^{-k}),$$

for all x, y with (35). Since k is arbitrary in (35), (14) follows, and Theorem 1 is proved.

4. Proof of Theorem 2

It is obvious that (25) will be proved if we construct such a function f, for which the value of upper limit in (25) is positive. Furthermore, we may assume that

$$\psi(\xi)\geqslant\log\frac{e}{\xi}\qquad(0<\xi\leqslant 2). \tag{36}$$

Let N denote the set of nonnegative integers, and let

$$\begin{aligned} a_k&=\left(\frac{\psi(2)}{\psi(2\cdot 4^{-k-1})}\right)^{1/p},\\ r_k&=\min\{r\in N\colon\ 2^{-r}\leqslant\delta_k\},\\ s_k&=\max\{s\in N\colon\ s2^{-r_k}\leqslant a_k^p\}\qquad(k\in N)\end{aligned} \tag{37}$$

where $\{\delta_k\}$ is defined by (8). Then (24) implies that

$$\sum_{k=0}^{\infty}s_k2^{-r_k}=\infty. \tag{38}$$

In fact, if we observe (10), we see that $r_k\geqslant k$ $(k\in N)$, and so, by (37),

$$\sum_{k=0}^{\infty}s_k2^{-r_k}\geqslant\sum_{k=0}^{\infty}(s_k+1)2^{-r_k}-2\geqslant\sum_{k=0}^{\infty}\frac{\psi(2)}{\psi(2\cdot 4^{-k-1})}-2=\infty.$$

Furthermore, let

$$K=\{k\in N\colon\ a_k^{-p}\leqslant k^2\}. \tag{39}$$

Then, according to the definition of a_k, r_k, and s_k,

$$\sum_{k\notin K}s_k2^{-r_k}\leqslant\sum_{k\notin K}a_k^p<\sum_{k=1}^{\infty}1/k^2<\infty$$

and thus, by (38),

$$\sum_{k\in K}s_k2^{-r_k}=\infty. \tag{40}$$

Notice that it follows from (36) that the assumption

$$\sum_{k_1\leqslant k\leqslant k_2}s_k2^{-r_k}\geqslant 1 \tag{41}$$

implies that if $k_1\geqslant k_0$, where k_0 is sufficiently large, then

$$k_2\geqslant 2k_1. \tag{42}$$

Now we construct the function, for which the assertions of Theorem 2 are valid. In what follows we consider the points x, and the sets of these points, which coincide mod 1, as identical. All the functions will have period 1.

Write

$$S_k=\{x\colon\ x=s\cdot 2^{-r_k},\ 0\leqslant s\leqslant s_k\},\qquad I_k=[0,s_k2^{-r_k}) \tag{43}$$

and define the polygonal $f_k(x)$ as follows:

$$f_k(x)=\begin{cases}2^{r_k}a_k^{-1}\omega(\delta_k)\,\mathrm{dist}(x,S_k), & \text{if}\quad x\in I_k;\\ 0, & \text{if}\quad x\notin I_k.\end{cases} \tag{44}$$

It is obvious that f_k is absolutely continuous,

$$f_k(x)=0\quad(x\notin I_k);\qquad |f_k(x)|\leqslant\tfrac{1}{2}a_k^{-1}\omega(\delta_k)\quad(x\in I_k), \tag{45}$$

and, if we ignore the set S_k,

(46) $$f_k'(x) = 0 \quad (x \notin I_k); \quad |f_k'(x)| = 2^{r_k} a_k^{-1}\omega(\delta_k) \quad (x \in I_k).$$

Estimates (45) and (46) (see also (37)) imply in particular that

(47) $$\omega_\infty(f_k, \delta) \leqslant 2a_k^{-1}\min\left(\frac{\omega(\delta_k)}{\delta_k}\delta, \omega(\delta_k)\right).$$

Furthermore, compute the L_p-modulus of continuity of f_k. Since

$$\omega_p(f_k, \delta) \leqslant \omega_\infty(f_k, \delta)\,(2\operatorname{meas}(\operatorname{supp} f_k \cap [0,1]))^{1/p} = \omega_\infty(f, \delta)\,(2s_k 2^{-r_k})^{1/p},$$

we obtain from (47) and (37)

(48) $$\omega_p(f_k, \delta) \leqslant 4\min\left(\frac{\omega(\delta_k)}{\delta_k}\delta, \omega(\delta_k)\right).$$

Define the numerical sequence $\{\alpha_k\}_{k\in K}$ and the function $f(x)$ as follows:

(49) $$\alpha_k = \sum_{l\in K, l<k} s_l 2^{-r_l}, \qquad f(x) = \tfrac{1}{32}\sum_{k\in K} f_k(x-\alpha_k).$$

Using (48) and the estimate on the right-hand of (6) (with $c = 8$), we see that

$$\omega_p(f, \delta) \leqslant \frac{1}{32}\sum_{k\in K}\omega_p(f_k, \delta) \leqslant \frac{1}{8}\sum_{k=0}^{\infty}\min\left(\frac{\omega(\delta_k)}{\delta_k}\delta, \omega(\delta_k)\right) \leqslant \omega(\delta)$$

$$(0 \leqslant \delta \leqslant \tfrac{1}{2}),$$

and thus $f \in H_{p,\omega}$.

Let $l \in K$;

(50) $$G_l(x) = \tfrac{1}{32}\sum_{k\in K, k<l} f_k(x-\alpha_k), \qquad H_l(x) = \tfrac{1}{32}\sum_{k\in K,\, k>l} f_k(x-\alpha_k);$$

$$J_l = [\alpha_l, \alpha_l + s_l 2^{-r_l}] = [\alpha_l, \alpha_{l+1}),$$

(51) $$j_{l,q} = \left[\alpha_l + \frac{q}{2}2^{-r_l}, \alpha_l + \frac{q+1}{2}2^{-r_l}\right] \quad (q = 0, 1, \ldots, 2s_l - 1).$$

Notice that, according to (45) and (46), the supports of the functions $f_l(x-\alpha_l)$ and $f_l'(x-\alpha_l)$ are contained in J_l (mod 1). Next we prove that

(52) $$\limsup_{l\to\infty}\sup_{x\in J_l}|G_l'(x)|\frac{\delta_l}{\omega(\delta_l)} = 0,$$

(53) $$\limsup_{l\to\infty}\sup_{x\in J_l}\frac{|H_l(x)|}{\omega(\delta_{l+1})} = 0.$$

For x fixed, define the set of integers $K_x \subset K$ as follows:

$$K_x = \{k \in K\colon x \in J_k \pmod 1\}.$$

This set is an increasing sequence of integers $\{k_i\}_{i=0}^{\infty} \subset K$; (40) implies that K_x is infinite, and it follows from (41) that

(54) $$k_{i+1} \geqslant 2k_i \quad (i = 0, 1, \ldots).$$

Furthermore, if $x \in J_l$, then

(55) $$l \in K_x,$$

and thus we deduce from (45), (46), (8) (see also (39)):

$$32|G_l'(x)| \leqslant \sum_{k\in K_x, k<l}|f_k'(x-\alpha_k)| = \sum_{k\in K_x, k<l} 2^{r_k}a_k^{-1}\omega(\delta_k)$$

$$\leqslant 2a_l^{-1}\sum_{k\in K_x, k<l}\frac{\omega(\delta_k)}{\delta_k} \leqslant 2l^2\frac{\omega(\delta_l)}{\delta_l}\sum_{k\in K_x, k<l}4^{k-l},$$

$$32|H_l(x)| \leqslant \sum_{k\in K_x, k>l}|f_k(x-\alpha_k)| \leqslant \frac{1}{2}\sum_{k\in K_x, k>l}a_k^{-1}\omega(\delta_k)$$

$$\leqslant \frac{1}{2}\sum_{k\in K_x, k>l}k^2\omega(\delta_k) \leqslant \frac{\omega(\delta_{l+1})}{2}\sum_{k\in K_x, k>l}k^2 4^{l-k+1}$$

which, together with (54) and (55), gives (52) and (53), the rate of convergence to 0 being like that of a geometric progression.

Furthermore, (52), (53), and (50) imply that, if we define

(56) $$\varphi_l(x) = f(x) - f_l(x-\alpha_l),$$

then

(57) $$\limsup_{l\to\infty}\left\{\frac{|\varphi_l(x_1)-\varphi_l(x_2)|}{\omega(\delta_l)} : x_1, x_2 \in J_l, |x_1 - x_2| \leqslant \delta_l\right\} = 0.$$

Now we complete the proof of Theorem 2. Let F be a measurable set, and let x be a point of density for F. By the definition, there exists an infinite subsequence L_x of the sequence K_x, such that

(58) $$\operatorname{meas} F\cap J_l \geqslant \tfrac{3}{4}\operatorname{meas} J_l = \tfrac{3}{4}s_l 2^{-r_l} \qquad (l \in L_x).$$

Since by (51),

$$F\cap J_l = \bigcup_{q=0}^{2s_l-1} F\cap j_{l,q},$$

it follows from (58) that, for each $l \in L_x$, there is a q_l such that

(59) $$\operatorname{meas} F\cap j_{l,q_l} \geqslant \tfrac{3}{4}\operatorname{meas} j_{l,q_l} = \tfrac{3}{8}2^{-r_l} \geqslant \tfrac{3}{16}\delta_l > \delta_{l+2}.$$

The function $f_l(x-\alpha_l)$ is linear in each $j_{l,q}$, and we deduce from (59) that, for an arbitrary $l \in L_x$, there exists a pair of points $x_1^{(l)}, x_2^{(l)}$ such that

$$\tfrac{3}{16}\delta_l \leqslant |x_1^{(l)} - x_2^{(l)}| \leqslant \tfrac{1}{2}\delta_l; \qquad x_1^{(l)} \in F, \qquad x_2^{(l)} \in F;$$

(60) $$|f_l(x_1^{(l)}-\alpha_l) - f_l(x_2^{(l)}-\alpha_l)| \geqslant \frac{3}{8}a_l^{-1}\omega(\delta_l) = \frac{3}{8}\left(\frac{\psi(2\cdot 4^{-l-1})}{\psi(2)}\right)^{1/p}\omega(\delta_l)$$

$$\geqslant \tfrac{3}{8}\psi^{-1/p}(2)\,\omega(|x_1^{(l)}-x_2^{(l)}|)\,\psi^{1/p}(\Omega(|x_1^{(l)}-x_2^{(l)}|)).$$

If we observe also (56) and (57), we see from (60) that

$$(61)\qquad \overline{\lim_{\substack{x_1\to x,\, x_2\to x\\ x_1, x_2 \in F}}} \frac{|f(x_1)-f(x_2)|}{\omega(|x_2-x_1|)\psi^{1/p}(\Omega(|x_2-x_1|))} > 0,$$

which completes the proof of Theorem 2.

Remark. Slightly more accurate observations show that (61) holds at an arbitrary point x, which is a point of positive upper density for the set F.

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PROPERTIES OF BOUNDED ORTHOGONAL SPLINE BASES

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1. Introduction

The following system of spline functions defined by the equalities

$$(1)\qquad \begin{gathered} w_{-m}^{(m)}(t) = f_{-m}^{(m)}(t),\ \ldots,\ w_1^{(m)}(t) = f_1^{(m)}(t), \\ w_{2^\mu+l}^{(m)}(t) = \sum_{s=1}^{2^\mu} A_{ls}^{(\mu)} f_{2^\mu+s}^{(m)}(t), \quad 1 \leqslant l \leqslant 2^\mu, \quad \mu \geqslant 0,\ m \geqslant -1 \end{gathered}$$

is considered. For $m \geqslant -1$ the functions $w_n^{(m)}$ are uniformly bounded in n, $n \geqslant -m$. The system $\{f_n^{(m)}, n \geqslant -m\}$ of spline functions of order m, $m \geqslant -1$, is defined for an arbitrary $m \geqslant -1$ in [3]. For $m = -1$ it is a Haar system and for $m = 0$ it is a Franklin system. The functions $w_n^{(m)}$ for $m = -1$ form a Walsh system; for $m = 0$ the system $w_n^{(0)} = c_n$ has been considered by Z. Ciesielski in [2].

The matrix $A_{ls}^{(\mu)}$ is common for all m. It is defined by the connection (1) between Walsh and Haar systems (the case $m = -1$).

We show that some results of Z. Ciesielski [2] may be generalized to the case of an arbitrary m, $m \geqslant -1$ (cf. Theorems 1, 5, 6, 7, 10, 11, 12). Moreover, some new results are proved (see Theorems 3, 4, 8, 13).

We prove that each of the systems $\{w_n^{(m)}, n \geqslant -m\}$, $m \geqslant -1$, is a basis in $L_p(I)$ for $1 < p < \infty$. A generalization of this fact is obtained for systems $\{D^k w_n^{(m)}, n \geqslant k-m\}$, $\{H^k w_n^{(m)}, n \geqslant k-m\}$, $0 \leqslant k \leqslant m+1$, $m \geqslant -1$, k-times differentiated and, correspondingly integrated, where D is the differentiation operator and $Hf(t) = \int_t^1 f(u)\,du$.

In Section 6 we observe that these systems are Riesz bases in $L_2(I)$.

2. Preliminaries and notation

We assume that all the functions considered below are defined on the interval $I = \langle 0, 1\rangle$.