If we observe also (56) and (57), we see from (60) that

$$\overline{\lim_{\substack{x_1\to x,\, x_2\to x\\ x_1,x_2\in F}}} \frac{|f(x_1)-f(x_2)|}{\omega(|x_2-x_1|)\psi^{1/p}(\Omega(|x_2-x_1|))} > 0, \tag{61}$$

which completes the proof of Theorem 2.

Remark. Slightly more accurate observations show that (61) holds at an arbitrary point x, which is a point of positive upper density for the set F.

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PROPERTIES OF BOUNDED ORTHOGONAL SPLINE BASES

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1. Introduction

The following system of spline functions defined by the equalities

$$\begin{gathered} w_{-m}^{(m)}(t) = f_{-m}^{(m)}(t), \ \ldots, \ w_1^{(m)}(t) = f_1^{(m)}(t), \\ w_{2^\mu+l}^{(m)}(t) = \sum_{s=1}^{2^\mu} A_{ls}^{(\mu)} f_{2^\mu+s}^{(m)}(t), \quad 1 \leqslant l \leqslant 2^\mu, \quad \mu \geqslant 0,\ m \geqslant -1 \end{gathered} \tag{1}$$

is considered. For $m \geqslant -1$ the functions $w_n^{(m)}$ are uniformly bounded in n, $n \geqslant -m$. The system $\{f_n^{(m)}, n \geqslant -m\}$ of spline functions of order m, $m \geqslant -1$, is defined for an arbitrary $m \geqslant -1$ in [3]. For $m = -1$ it is a Haar system and for $m = 0$ it is a Franklin system. The functions $w_n^{(m)}$ for $m = -1$ form a Walsh system; for $m = 0$ the system $w_n^{(0)} = c_n$ has been considered by Z. Ciesielski in [2].

The matrix $A_{ls}^{(\mu)}$ is common for all m. It is defined by the connection (1) between Walsh and Haar systems (the case $m = -1$).

We show that some results of Z. Ciesielski [2] may be generalized to the case of an arbitrary m, $m \geqslant -1$ (cf. Theorems 1, 5, 6, 7, 10, 11, 12). Moreover, some new results are proved (see Theorems 3, 4, 8, 13).

We prove that each of the systems $\{w_n^{(m)}, n \geqslant -m\}$, $m \geqslant -1$, is a basis in $L_p(I)$ for $1 < p < \infty$. A generalization of this fact is obtained for systems $\{D^k w_n^{(m)}, n \geqslant k-m\}$, $\{H^k w_n^{(m)}, n \geqslant k-m\}$, $0 \leqslant k \leqslant m+1$, $m \geqslant -1$, k-times differentiated and, correspondingly integrated, where D is the differentiation operator and $Hf(t) = \int_t^1 f(u)\,du$.

In Section 6 we observe that these systems are Riesz bases in $L_2(I)$.

2. Preliminaries and notation

We assume that all the functions considered below are defined on the interval $I = \langle 0, 1\rangle$.

The *Haar functions* are defined as follows:

$$\chi_1(t) = 1 \quad \text{for} \quad t \in I,$$

(2)
$$\chi_{2^\mu+\nu}^{(t)} = \begin{cases} \sqrt{2^\mu} & \text{for} \quad t \in \left\langle \frac{\nu-1}{2^\mu}, \frac{2\nu-1}{2^{\mu+1}} \right), \\ -\sqrt{2^\mu} & \text{for} \quad t \in \left\langle \frac{2\nu-1}{2^{\mu+1}}, \frac{\nu}{2^\mu} \right), \\ 0 & \text{elsewhere in } \langle 0, 1 \rangle, \end{cases}$$

$\mu \geqslant 0$, $1 \leqslant \nu \leqslant 2^\mu$.

By $\{r_n(t),\ n \geqslant 1\}$ we denote the *Rademacher orthonormal system*, i.e.

$$r_n(t) = \frac{1}{\sqrt{2^{n-1}}}\left(\chi_{2^{n-1}+1}(t) + \ldots + \chi_{2^n}(t)\right).$$

The *Walsh functions* are defined as follows:

$$w_1(t) = 1 \quad \text{and} \quad w_n(t) = r_{n_1+1}(t) \ldots r_{n_i+1}(t)$$

whenever $n = 1 + 2^{n_1} + 2^{n_2} + \ldots + 2^{n_i}$ with $0 \leqslant n_1 < \ldots < n_i$.

For a given $\mu \geqslant 0$, $A_{sl}^{(\mu)}$ denotes the *Walsh matrix* defined by

(3)
$$A_{sl}^{(\mu)} = (w_{2^\mu+s}, \chi_{2^\mu+l}),$$

where $(f, g) = \int_0^1 f(u) g(u)\, du$ and $1 \leqslant l \leqslant 2^\mu$, $1 \leqslant s \leqslant 2^\mu$. For each $\mu \geqslant 0$, the Walsh matrix $A_{sl}^{(\mu)}$ is orthogonal, symmetric and satisfies the relation

(4)
$$A_{sl}^{(\mu)} = 2^{-\mu/2} w_s \left(\frac{2l-1}{2^{\mu+1}} \right).$$

By m we denote an integer parameter $m \geqslant -1$, and by G the operator $Gf(t) = \int_0^t f(u)\, du$.

Applying the Schmidt orthonormalization procedure to the set of functions $\{1, t, t^2, \ldots, t^{m+1}, G^{m+1}\chi_n,\ n = 2, 3, \ldots\}$, we get a complete orthonormal system $\{f_n^{(m)},\ n \geqslant -m\}$ (for its properties see [4]).

Define, for $0 \leqslant k \leqslant m+1$ and $n \geqslant k-m$,

(5)
$$f_n^{(m,k)} = D^k f_n^{(m)}, \quad g_n^{(m,k)} = H^k f_n^{(m)}$$

where D and H are the differentiation operator d/dt and the integration operator $\int_t^1$, respectively.

The set $\{f_i^{(m,k)}, g_j^{(m,k)},\ i, j \geqslant k-m\}$ is biorthogonal and for $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$ it is a basis in $L_p(I)$ for $1 \leqslant p < \infty$.

The *modulus of smoothness of order* $r \geqslant 1$ *of the function* $f \in L_p(I)$ is defined for

finite p and $\delta r \leqslant 1$ by the formula

$$\omega_r^{(p)}(f, \delta) = \sup_{0 < h \leqslant \delta} \left(\int_0^{1-rh} |\Delta_h^r f(t)|^p\, dt \right)^{1/p}$$

and for $p = \infty$ by the formula

$$\omega_r^{(\infty)}(f, \delta) = \sup_{h \leqslant \delta} \{ |\Delta_h^r f(t)|,\ 0 \leqslant t < t + rh \leqslant 1 \},$$

where the symbol Δ_h^r denotes the forward progressive difference of order r with increment h.

The space $\mathrm{Lip}(\alpha, r, p)$, $0 < \alpha < r$, is defined as the set of functions belonging to $L_p(I)$ for $1 \leqslant p < \infty$ and to $C(I)$ for $p = \infty$ for which we have

(6)
$$\omega_r^{(p)}(f, \delta) = O(\delta^\alpha)$$

as $\delta \to 0$.

3. Bounded spline bases

For a given $m \geqslant -1$ we define the system $\{w_n^{(m)},\ n \geqslant -m\}$ as follows:

(7)
$$w_{-m}^{(m)}(t) = f_{-m}^{(m)}(t), \quad \ldots, \quad w_0^{(m)}(t) = f_0^{(m)}(t), \quad w_1^{(m)}(t) = f_1^{(m)}(t),$$
$$w_{2^\mu+\nu}^{(m)}(t) = \sum_{l=1}^{2^\mu} A_{\nu l}^{(\mu)} f_{2^\mu+l}^{(m)}(t),$$

where $1 \leqslant \nu \leqslant 2^\mu$, $\mu \geqslant 0$.

It is clear that

(8)
$$f_{2^\mu+\nu}^{(m)}(t) = \sum_{l=1}^{2^\mu} A_{\nu l}^{(\mu)} w_{2^\mu+l}^{(m)}(t).$$

In the case of $m = -1$ the functions $w_n^{(m)}$, $n \geqslant 1$, are Walsh functions and in the case of $m = 0$ they are the functions c_n, $n \geqslant 0$, constructed by Z. Ciesielski [2].

Let us define for $0 \leqslant k \leqslant m+1$ and $n \geqslant k-m$

(9)
$$w_n^{(m,k)} = D^k w_n^{(m)},$$

(10)
$$u_n^{(m,k)} = H^k w_n^{(m)}.$$

The definition of a Walsh matrix gives

(11)
$$(w_i^{(m,k)}, u_j^{(m,k)}) = \delta_{ij}, \quad i, j \geqslant k-m,$$

i.e. the set $\{w_i^{(m,k)}, u_j^{(m,k)},\ i, j \geqslant k-m\}$, $0 \leqslant k \leqslant m+1$, is biorthogonal. It is assumed in this notation that $w_n^{(m,0)} = w_n^{(m)}$, $u_n^{(m,0)} = w_n^{(m)}$.

THEOREM 1. *Let* $m \geqslant -1$. *Then the system* $\{w_n^{(m)},\ n \geqslant -m\}$ *is a bounded, orthonormal and complete system in* $L_2(I)$ *such that*

(12)
$$(w_i^{(m)}, f_j^{(m)}) = (w_i, \chi_j), \quad i, j > 0.$$

The proof of this theorem is based on Theorem 2.2 of [4] and runs analogously to that of Theorem 1 in [2]; thus we omit the details.

Using Theorem 6.1 of [4], we get

THEOREM 2. *Let $m \geqslant -1$, $0 \leqslant k \leqslant m+1$. Then there is a constant M_m depending only on m and such that*

$$|w_{2^\mu+l}^{(m,k)}(t)| \leqslant M_m 2^{\mu k}, \tag{13}$$

$$|u_{2^\mu+l}^{(m,k)}(t)| \leqslant M_m 2^{-\mu k}, \tag{14}$$

where $1 \leqslant l \leqslant 2^\mu$, $\mu \geqslant 0$.

In what follows we use the following notation. For $f \in L_1(I)$, let

$$W_n^{(m,k)}(f) = \sum_{j=k-m}^{n} (f, u_j^{(m,k)}) w_j^{(m,k)}, \tag{15}$$

$$P_n^{(m,k)}(f) = \sum_{j=k-m}^{n} (f, g_j^{(m,k)}) f_j^{(m,k)}, \tag{16}$$

$$W_n^{(m,0)}(f) = W_n^{(m)}(f), \quad P_n^{(m,0)}(f) = P_n^{(m)}. \tag{17}$$

It should be clear that the definition of the system $\{w_n^{(m,k)}, n \geqslant k-m\}$ implies

$$W_{2^\mu}^{(m,k)}(f) = P_{2^\mu}^{(m,k)}(f) \quad 0 \leqslant k \leqslant m+1,\ \mu \geqslant 0,\ m \geqslant -1. \tag{18}$$

THEOREM 3. *Let m and k be fixed: $0 \leqslant k \leqslant m+1$, $m \geqslant -1$. Then the system $\{w_n^{(m,k)}, n \geqslant k-m\}$ is a basis in $L_p(I)$ with $1 < p < \infty$ and for f in $L_p(I)$ we have*

$$f = \sum_{n=k-m}^{\infty} (f, u_n^{(m,k)}) w_n^{(m,k)}. \tag{19}$$

Proof. It is clear that the theorem will be proved if we show the validity of the following inequalities:

$$\Big\|\sum_{n=2^\mu+1}^{2^\mu+\nu} b_n w_n^{(m,k)}\Big\|_p \leqslant c_{m,p}\Big\|\sum_{n=2^\mu+1}^{2^{\mu+1}} b_n w_n^{(m,k)}\Big\|_p, \quad \mu \geqslant 0, \quad 1 \leqslant \nu \leqslant 2^\mu, \tag{20}$$

for all sequences $\{b_n\}$ of real numbers, where $1 < p < \infty$ and the factor $c_{m,p}$ depends on m and p only.

Let $\{b_n\}$ be a given real sequence and $1 \leqslant \nu \leqslant 2^\mu$, $\mu \geqslant 0$. Then, using the inequality of Ciesielski (Theorem 7.1 in [4]), we obtain the estimates

$$\begin{aligned}\Big\|\sum_{n=2^\mu+1}^{2^\mu+\nu} b_n w_n^{(m,k)}\Big\|_p &= \Big\|\sum_{i=2^\mu+1}^{2^{\mu+1}} a_{i,\nu} f_i^{(m,k)}\Big\|_p \\ &\leqslant M_1 2^{\mu k} 2^{\mu(1/2-1/p)}\Big(\sum_{i=2^\mu+1}^{2^{\mu+1}} |a_{i,\nu}|^p\Big)^{1/p},\end{aligned} \tag{21}$$

where

$$a_{2^\mu+l,\nu} = \sum_{j=1}^{\nu} A_{lj}^{(\mu)} b_{2^\mu+j}.$$

For an arbitrary real sequence $\{\alpha_n\}$ we have the equality

$$\Big\|\sum_{n=2^\mu+1}^{2^{\mu+1}} \alpha_n \chi_n\Big\|_p = 2^{\mu(1/2-1/p)}\Big(\sum_{n=2^\mu+1}^{2^{\mu+1}} |\alpha_n|^p\Big)^{1/p}, \quad \mu \geqslant 0. \tag{22}$$

This and the results of Paley [6] imply

$$\begin{aligned}2^{\mu(1/2-1/p)}\Big(\sum_{i=2^\mu+1}^{2^{\mu+1}} |a_{i,\nu}|^p\Big)^{1/p} &= \Big\|\sum_{i=2^\mu+1}^{2^{\mu+1}} a_{i,\nu}\chi_i\Big\|_p = \Big\|\sum_{j=2^\mu+1}^{2^\mu+\nu} b_j w_j\Big\|_p \\ &\leqslant M_2\Big\|\sum_{j=2^\mu+1}^{2^{\mu+1}} b_j w_j\Big\|_p = M_2\Big\|\sum_{i=2^\mu+1}^{2^{\mu+1}} a_i\chi_i\Big\|_p,\end{aligned} \tag{23}$$

where

$$a_{2^\mu+l} = \sum_{j=1}^{2^\mu} A_{lj}^{(\mu)} b_{2^\mu+j}.$$

Combining (21), (23) and (22), and using the inequality of Ciesielski, we complete the proof. A similar argument gives

THEOREM 4. *Let $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$. Then $\{u_n^{(m,k)}, n \geqslant k-m\}$ is a basis in $L_p(I)$ for $1 < p < \infty$, and for g in $L_p(I)$ we have*

$$g = \sum_{j=k-m}^{\infty} (g, w_j^{(m,k)}) u_j^{(m,k)}. \tag{24}$$

LEMMA 1. *If f is a function of bounded variation on $\langle 0, 1\rangle$, then, for each $m \geqslant -1$, there exists a constant M_m such that*

$$2^{\mu k}\sum_{n=2^\mu+1}^{2^{\mu+1}} |(f, g_n^{(m,k)})| \leqslant M_m \operatorname*{var}_I f, \tag{25}$$

for $0 \leqslant k \leqslant m+1$ and $\mu \geqslant 0$.

This follows from the inequality of Ciesielski, Theorem 4.1 of [4] and from the property of the L_1 modulus of smoothness (cf. [1], p. 52).

THEOREM 5. *Let $m \geqslant -1$. Then there are constants M_m, B_m such that the inequalities*

$$0 < M_m n \leqslant \operatorname*{var}_I w_n^{(m)} \leqslant B_m n \tag{26}$$

hold for $n > 0$.

This theorem can be proved with the aid of Lemma 1 in the same way as Corollary 1 of [2].

THEOREM 6. *Let $0 \leqslant k \leqslant m+1$, $m \geqslant -1$ and $1 \leqslant p \leqslant \infty$. Then the inequalities*

$$\|w_n^{(m,k)}\|_p \sim n^k, \quad n > 0,(^1) \tag{27}$$

$$\|u_n^{(m,k)}\|_p \sim n^{-k}, \quad n > 0 \tag{28}$$

hold uniformly in k and p.

---

(1) The symbol $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$.

The proof follows from the definitions $\{w_n^{(m,k)}, n \geqslant k-m\}$, $\{u_n^{(m,k)}, n \geqslant k-m\}$ with the aid of Ciesielski's inequalities.

4. Lebesgue constants and approximation

Let $L_n^{(m)}$ denote the Lebesgue constants for the system $\{w_n^{(m)}, n \geqslant -m\}$.

THEOREM 7. *The Lebesgue constants of the orthonormal systems* $\{w_n^{(m)}, n \geqslant -m\}$ *satisfy the following inequalities*:

$$L_{2^n}^{(m)} = O(1), \quad L_n^{(m)} = O(\log n), \tag{29}$$

where the factors in O depend on m only.

The proof of the first part is clear. The second part is based on the first part and on the estimates of the Lebesgue constants for the Walsh system (for details see Theorem 2 of [2]).

The best approximation in $L_p(I)$ for $1 \leqslant p < \infty$ and in $C(I)$ for $p = \infty$ is defined as follows:

$$E_{n,p}^{(m-k)}(f)_W = \inf_{\{a_{-m},\ldots,a_n\}} \|f - T_n^{(m,k)}\|_p, \tag{30}$$

$$T_n^{(m,k)} = \sum_{j=k-m}^{n} a_j w_j^{(m,k)}, \quad 0 \leqslant k \leqslant m+1. \tag{31}$$

Now, let $E_{n,p}^{(m-k)}(f)$ stand for the best approximation by polynomials

$$U_n^{(m,k)} = \sum_{j=k-m}^{n} b_j f_j^{(m,k)}.$$

It is clear by (18) that, for each $n > 0$,

$$E_{2^n,p}^{(m)}(f) = E_{2^n,p}^{(m)}(f)_W. \tag{32}$$

THEOREM 8. *Let* $m \geqslant -1$, $f \in L_p(I)$ *for* $1 < p < \infty$. *Then there is a constant* $M_{m,p}$ *depending on m and p only and such that the inequalities*

$$E_{n,p}^{(m-k)}(f)_W \leqslant \|f - W_n^{(m,k)} f\|_p \leqslant M_{m,p} E_{n,p}^{(m-k)}(f)_W,$$
$$\|f - W_n^{(m,k)} f\|_p \leqslant M_{m,p}\,\omega_{m-k+2}^{(p)}(f, 1/n) \tag{33}$$

hold for $n \geqslant m-k+2$, $0 \leqslant k \leqslant m+1$.

Theorem 8 is a consequence of Theorem 3 and Theorem 4.1 of [4], and of the properties of best approximation.

In the case $p = 1$ and $p = \infty$ we have the following estimates:

$$\|f - W_n^{(m)} f\|_1 \leqslant M_m \log n\, E_{n,1}^{(m)}(f)_W, \tag{34}$$

$$\|f - W_n^{(m)} f\|_\infty \leqslant M_m \log n\, E_{n,\infty}^{(m)}(f)_W. \tag{35}$$

The proof of these inequalities is standard and it is based on Theorem 7.

COROLLARY 1. *Let* $m \geqslant -1$, $f \in L_p(I)$ *for* $1 \leqslant p < \infty$ *and* $f \in C(I)$ *for* $p = \infty$.

Then there are constants $M_{m,p}$ *and* B_m *such that the estimate*

$$\|f - W_n^{(m)} f\|_p \leqslant M_{m,p}\,\omega_{m+2}^{(p)}(f, 1/n) \tag{36}$$

holds for $n \geqslant m+2$, $1 < p < \infty$ *and the estimates*

$$\|f - W_n^{(m)} f\|_1 \leqslant M_m \log n\, \omega_{m+2}^{(1)}(f, 1/n), \tag{37}$$

$$\|f - W_n^{(m)} f\|_\infty \leqslant M_m \log n\, \omega_{m+2}^{(\infty)}(f, 1/n) \tag{38}$$

holds for $n \geqslant m+2$.

Condition (32) and Theorem 9.1 of [4] imply the following assertion:

THEOREM 9. *Let* m, p *and* f *be given such that* $m \geqslant -1$, $1 \leqslant p \leqslant \infty$, $f \in L_p(I)$ *if* $1 \leqslant p < \infty$, *and* $f \in C(I)$ *in the case of* $p = \infty$. *Then there exists a constant* M_m *such that the inequalities*

$$\omega_k^{(p)}\left(f, \frac{1}{n}\right) \leqslant \frac{M_m}{n^k}\Big(\|f\|_p + \sum_{i=m+2}^{n} i^{k-1} E_{i,p}^{(m)}(f)_W\Big), \tag{39}$$

and

$$\omega_{m+2}^{(m)}\left(f, \frac{1}{n}\right) \leqslant \frac{M_m}{n^{m+1+1/p}}\Big(\|f\|_p + \sum_{i=m+2}^{n} i^{m+1/p} E_{i,p}^{(m)}(f)_W\Big) \tag{40}$$

hold for $n \geqslant m+2$ *and* $1 \leqslant k \leqslant m+1$.

Inequalities (42) in the case of $m = -1$ were proved in [5].

5. Fourier coefficients and Lipschitz classes

THEOREM 10. *Let* $f \in L_1(I)$. *There exists a constant* M_m *such that the inequality*

$$|(f, u_n^{(m,k)})| \leqslant M_m \frac{1}{n^k}\,\omega_{m-k+2}^{(1)}\left(f, \frac{1}{n}\right) \tag{41}$$

holds for $n \geqslant m-k+2$, $0 \leqslant k \leqslant m+1$.

The proof of Theorem 10 follows from the definition of the system $\{u_n^{(m,k)}, n \geqslant k-m\}$ and from the inequalities of Ciesielski.

COROLLARY 2. *If f is a function of bounded variation on* $\langle 0, 1\rangle$, *then*

$$|(f, u_n^{(m,k)})| = O(1/n^{k+1}), \quad n > 0,\ 0 \leqslant k \leqslant m+1.$$

COROLLARY 3. *If* $D^{m+1} f$ *is a function of bounded variation on* $\langle 0, 1\rangle$, *then*

$$|(f, w_n^{(m)})| = O(1/n^{m+2}), \quad n > 0.$$

THEOREM 11. *Let* $m \geqslant 0$, $0 < \alpha < m+1$, $1 \leqslant p < \infty$. *Then*

$$\sum_{n=-m}^{\infty} b_n w_n^{(m)}(t) \tag{42}$$

converges in $L_p(I)$ *and it is a Fourier series of the function* $f \in \mathrm{Lip}(\alpha, m+1, p)$ *if and* *only if*

$$b_{2^\mu+\nu} = 2^{-\mu(1/2+\alpha)} \sum_{l=1}^{2^\mu} A_{\nu l}^{(\mu)} a_{2^\mu+l}, \qquad 1 \leqslant \nu \leqslant 2^\mu, \tag{43}$$

$$\Big(2^{-\mu} \sum_{l=1}^{2^\mu} |a_{2^\mu+l}|^p\Big)^{1/p} = O(1).$$

THEOREM 12. *Let* $0 < \alpha < m+1$, $m \geqslant 0$ *and let the series* (42) *be lacunary. Then the series* (42) *is a Fourier series of* $f \in \mathrm{Lip}(\alpha, m+1, \infty)$ *if and only if* $b_n = 0(n^{-\alpha})$.

The proofs of these theorems are omitted since they closely follow in outline the proofs of Theorems 5 and 6 of [2].

6. Unconditional bases

Let $m \geqslant -1$, $0 \leqslant k \leqslant m+1$, f, $g \in L_2(I)$ and

$$f = \sum_{n=k-m}^{\infty} a_n f_n^{(m,k)}, \tag{44}$$

$$g = \sum_{n=k-m}^{\infty} b_n g_n^{(m,k)}. \tag{45}$$

It has been proved in [7] that the inequalities

$$\|f\|_2 \sim \Big(\sum_{n=k-m}^{\infty} (n^k a_n)^2\Big)^{1/2}, \tag{46}$$

$$\|g\|_2 \sim \Big(\sum_{n=k-m}^{\infty} \Big(\frac{b_n}{n^k}\Big)^2\Big)^{1/2} \tag{47}$$

hold for $0 \leqslant k \leqslant m+1$ (i.e. the systems $\{f_n^{(m,k)}, n \geqslant k-m\}$ and $\{g_n^{(m,k)}, n \geqslant k-m\}$ are unconditional bases in $L_2(I)$). Let $h_n^{(m,k)}$ denote

$$h_n^{(m,k)} = \begin{cases} \dfrac{w_n^{(m,k)}}{\|w_n^{(m,k)}\|_2} & \text{for} \quad 0 \leqslant k \leqslant m+1, \\[2ex] \dfrac{u_n^{(m,-k)}}{\|u_n^{(m,-k)}\|_2} & \text{for} \quad 0 \leqslant -k \leqslant m+1. \end{cases}$$

Using the inequalities (46), (47) and Theorem 6, we obtain

THEOREM 13. *Let* $m \geqslant -1$, $0 \leqslant |k| \leqslant m+1$. *Then the system of functions* $\{h_n^{(m,k)}, n \geqslant |k|-m\}$ *is a Riesz basis in* $L_2(I)$, *i.e. it is an unconditional basis in* $L_2(I)$.

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