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INEQUALITIES FOR FRACTIONAL DERIVATIVES ON THE HALF-LINE

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1

Let n be integers, S the real line $(-\infty, \infty)$ or the half-line $[0, \infty)$. Denote by $W_n(S)$ the set of functions $x \in C = C(S)$ such that $x^{(n)} \in L_\infty = L_\infty(S)$. It is known (G. H. Hardy, J. E. Littlewood [7], L. Neder [11]) that functions from $W_n(S)$ satisfy for $0 < k < n$ the inequality

$$\|x^{(k)}\|_C \leqslant G\|x\|_C^{1-k/n}\|x^{(n)}\|_{L_\infty}^{k/n}, \tag{1.1}$$

where $G = G(k, n, S)$. A. N. Kolmogorov [8] has calculated the least constant G in (1.1) on the whole line $S = (-\infty, \infty)$. A corresponding result for the half-line is known only for $n = 2$ (E. Landau [9], J. Hadamard [6]) and for $n = 3$ (A. P. Matorin [10]); in particular, the Landau–Hadamard inequality has the form

$$\|x'\|_C \leqslant 2\sqrt{\|x\|_C\,\|x''\|_{L_\infty}}. \tag{1.2}$$

Estimates of the constant G for $S = [0, \infty)$ were given in papers of A. Gornyr H. Cartan, A. P. Matorin, S. B. Stechkin and etc. (for references see [14]); pape. [12] of I. Schoenberg and Al. Cavaretta deals with the calculation of the best constant,

In this note inequality (1.1) is considered for derivatives of not necessarily integer order on the half-line $S = [0, \infty)$. The value of the least constant G is calculated here for small k, n; in the general case lower and upper estimates for G are given.

The notion of integrals and derivatives of arbitrary order has been introduced by J. Liouville, B. Riemann and in the periodic case by H. Weyl. The subject has been considered by G. H. Hardy, J. E. Littlewood, H. Kober and others; detailed references may be found in H. Bateman and A. Erdelyi [1].

One of the following two operators is called *integral of order* $\sigma > 0$:

$$(J_\sigma y)(t) = \frac{1}{\Gamma(\sigma)}\int_t^\infty (\eta - t)^{\sigma-1} y(\eta)\,d\eta, \tag{1.3}$$

$$(J_\sigma^* y)(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\eta)^{\sigma-1} y(\eta)\,d\eta. \tag{1.4}$$

By these formulas the operator J_σ is defined, for example, on the set $M \subset L_\infty$ of bounded functions, approaching 0 at infinity faster than any power of the argument; the operator J_σ^* is defined on the set $\mathscr{L}$ of locally summable functions. The operators J_σ, J_σ^* are adjoint and for any $\alpha > 0$, $\beta > 0$

$$J_\alpha J_\beta = J_{\alpha+\beta}, \qquad J_\alpha^* J_\beta^* = J_{\alpha+\beta}^*.$$

For $y \in M$ the function $J_\sigma y$ uniquely defines the function y (see [15], Theorem 153). Consequently, on M the operator J_σ has the inverse operator $D_\sigma = J_\sigma^{-1}$. In the case of integer σ we have $(J_\sigma y)^{(\sigma)} = (-1)^\sigma y$ and thus $D_\sigma x = (-1)^\sigma x^{(\sigma)}$ for $x \in J_\sigma M$. If a function x has a compact support and $x \in W_m(S)$ for some integer $m > \sigma$, then $x \in J_\sigma M$ and $D_\sigma x = (-1)^m J_{m-\sigma} x^{(m)}$, since $J_\sigma J_{m-\sigma} x^{(m)} = J_m x^{(m)} = (-1)^m x$. For $0 < \sigma < 1$, $m = 1$, we have (see [16], no. 9.8.1)

$$(D_\sigma x)(t) = \frac{\sigma}{\Gamma(1-\sigma)} \int_t^\infty (\eta - t)^{-\sigma-1}[x(t) - x(\eta)]\,d\eta. \tag{1.5}$$

By this formula the operator D_σ can be extended to a larger set of functions. Such an operator has been considered in papers [4], [5] by S. P. Geisberg. In particular, in [4] it has been proved that, if $S = (-\infty, \infty)$, $0 < s < \sigma < 1$, then there is a finite constant $G = G(S, \sigma)$ such that for any function $x \in C$ satisfying the Lipshitz condition of order $\gamma = \gamma(x) > \sigma$ the inequality

$$\|D_s x\|_C \leqslant G \|x\|_C^{1-s/\sigma} \|D_\sigma x\|_C^{s/\sigma}$$

holds. In [5] the best constant has been found in such an inequality in the case $0 < s < 1$, $\sigma = 2$ $(x \in W_2(-\infty, \infty),\ D_2 x = x'')$.

In what follows we shall also understand fractional differentiation as a certain extension of the operator D_σ, and the process of extending of the operator D_σ from the set $J_\sigma M$ to a wider set $W_\sigma \subset C$ will follow S. L. Sobolev's scheme. Everywhere below $S = [0, \infty)$.

Denote the inverse operator to J_σ^* by D_σ^*; if σ is a positive integer, then $D_\sigma^* x = x^{(\sigma)}$. Let K be the set of infinitely differentiable functions on $[0, \infty)$, vanishing outside of the interval $[a, b] \subset (0, \infty)$, the latter one depending on a function. It is not difficult to verify that $K \subset J_\sigma^* \mathscr{L}$ and for $\varphi \in K$

$$D_\sigma^* \varphi = J_{m-\sigma}^* \varphi^{(m)},$$

for any positive integer $m \geqslant \sigma$. If σ is an integer, then $D_\sigma^* K \subset K$, and if σ is not an integer, then for $\varphi \in K$

$$(D_\sigma^* \varphi)(t) = O(t^{-\sigma-1}) \quad \text{when} \quad t \to \infty,$$

therefore $D_\sigma^* K \subset L$ for any $\sigma > 0$.

The operator D_σ^* is an adjoint operator to D_σ, i.e.

$$\int_0^\infty x D_\sigma^* \varphi\,dt = \int_0^\infty \varphi D_\sigma x\,dt, \qquad \varphi \in K, \qquad x \in J_\sigma M.$$

We use this relation to extend the definition of the domain of the operator D_σ. Given a pair of functions $x \in C[0, \infty)$, $y \in L_\infty[0, \infty)$, we say that $x \in W_\sigma$ and $y = D_\sigma x$, if

$$\int_0^\infty x D_\sigma^* \varphi\,dt = \int_0^\infty \varphi y\,dt \qquad \forall \varphi \in K. \tag{1.6}$$

If σ is a positive integer, then, as is well known, (1.6) implies that x has a locally absolutely continuous derivative of order $\sigma - 1$ on $[0, \infty)$, $y = (-1)^\sigma x^{(\sigma)}$; thus W_σ coincides with the class $W_\sigma[0, \infty)$ introduced above.

It is plain that $J_\sigma M \subset W_\sigma$. Notice furthermore that if the function $x \in C$ possesses derivative $x^{(m)}$ of order $m \geqslant \sigma$ and $\int_1^\infty t^{m-\sigma-1}|x^{(m)}(t)|\,dt < \infty$, then $x \in W_\sigma$ and $D_\sigma x = (-1)^m J_{m-\sigma} x^{(m)}$; this fact may be verified directly, by proceeding from (1.6).

It will be proved below that, if $0 < s < \sigma$, then $W_\sigma \subset W_s$, $D_s W_\sigma \subset C$ and the modulus of continuity

$$\omega(\delta) = \sup_{\|x\|_C \leqslant \delta,\ x \in Q} \|D_s x\|_C \tag{1.7}$$

of the operator D_s, on the class $Q = \{x \in W_\sigma\colon \|D_\sigma x\|_{L_\infty} \leqslant 1\}$, is finite for any $\delta > 0$. Along with (1.7) we will consider a problem formulated by S. B. Stechkin [13] concerning the quantity

$$E(N) = \inf_{\|T\|_C^C \leqslant N} \sup_{x \in Q} \|D_s x - Tx\|_C \tag{1.8}$$

of the best approximation of the operator D_s by bounded linear operators T on the class Q. S. B. Stechkin [13] noticed that for any $N > 0$, $\delta > 0$

$$E(N) \geqslant \omega(\delta) - N\delta \tag{1.9}$$

and consequently

$$E(N) \geqslant \Omega(N) = \sup_{\delta > 0} \{\omega(\delta) - N\delta\}. \tag{1.10}$$

The transformation: $x(t) \to x_h(t) = x(ht)$ maps W_σ into itself while $(D_\gamma x_h)(t) = h^\gamma (D_\gamma x)(ht)$ for any $\gamma \leqslant \sigma$. This easily implies the validity of the formula (cf. [13], [2])

$$\omega(\delta) = G\delta^{\frac{\sigma-s}{\sigma}}, \qquad G = \omega(1). \tag{1.11}$$

If we insert it into (1.10), we obtain the inequality

$$E(N) \geqslant \Omega(N) = \frac{s}{\sigma}\left(\frac{\sigma - s}{\sigma}\right)^\nu G^{s/\sigma} N^{-\nu}, \qquad \nu = \frac{\sigma - s}{s}. \tag{1.12}$$

2

Along with (1.7), (1.8), we define the quantity

$$\varepsilon(N) = \inf_{\bigvee_0^\infty \zeta \leqslant N,\ \zeta(0) = 0} \|\theta - (J_\sigma^* \zeta)'\|_{L(0,\infty)} \tag{2.1}$$

of the best approximation of the fixed function

(2.2) $$\theta(t) = \frac{t^{\sigma-s-1}}{\Gamma(\sigma-s)}$$

by the functions $(J_\sigma^*\zeta)'$. In the case $\sigma \geqslant 1$ we have $(J_\sigma^*\zeta)' = J_{\sigma-1}^*\zeta$ and, if $0 < \sigma < 1$, then we confine our considerations in (2.1) only to functions ζ such that $J_\sigma^*\zeta$ is locally absolutely continuous. It will be shown below that $\varepsilon(N) = E(N) = \Omega(N)$.

Let us list some properties of the quantity $\varepsilon(N)$. For an arbitrary $h > 0$ the mapping $\zeta(t) \to \zeta_h(t) = h^s\zeta(ht)$ is one-to-one and

$$\bigvee \zeta_h = h^s \bigvee \zeta, \quad \|\theta-(J_\sigma^*\zeta_h)'\| = h^{s-\sigma}\|\theta-(J_\sigma^*\zeta)'\|.$$

Therefore, taking $h = N^{1/s}$, we get

(2.3) $$\varepsilon(N) = N^{-\nu}\varepsilon(1), \quad \nu = \frac{\sigma-s}{s}.$$

LEMMA 1. *The quantity $\varepsilon(N)$ is finite. Moreover, the following inequality holds*

(2.4) $$\varepsilon\left(\frac{2^{\sigma+1}\Gamma(\sigma)}{\Gamma(s+1)\Gamma(\sigma-s)}\right) \leqslant \frac{\Gamma(\sigma)}{\Gamma(s)\Gamma(\sigma-s+1)}$$

Proof. We first show that, for $a > \sigma-1$,

(2.5) $$t^a e^{-t} = J_\sigma^* t^{a-\sigma}e^{-t}\varrho(t),$$

where

(2.6) $$\varrho(t) = \sum_{m=0}^{\infty}\alpha_m(-t)^m, \quad \alpha_m = \frac{\Gamma(a+1)\Gamma(\sigma+1)}{\Gamma(\sigma-m+1)\Gamma(a-\sigma+m+1)\Gamma(m+1)}.$$

Notice that (2.5) can also be written in the form

(2.7) $$D_\sigma^* t^a e^{-t} = t^{a-\sigma}e^{-t}\varrho(t).$$

If the number σ is a positive integer, then in (2.6) only the first $\sigma+1$ terms do not vanish; in this case the formulas (2.6), (2.7) may be checked sufficiently easily.

The series (2.6) converges absolutely on the whole real line. Multiplying it by the series for e^{-t}, we get

$$e^{-t}\varrho(t) = \sum_{n=0}^{\infty}(-t)^n\sum_{k=0}^{n}\alpha_k\frac{1}{(n-k)!}.$$

Transform the coefficients of this series. We have

$$(t^{a+n})^{(n)} = t^a\frac{\Gamma(a+n+1)}{\Gamma(a+1)},$$

and on the other hand

$$(t^{a+n})^{(n)} = (t^\sigma t^{a+n-\sigma})^{(n)} = \sum_{k=0}^{n} C_n^k(t^\sigma)^{(k)}(t^{a+n-\sigma})^{(n-k)}$$

$$= t^a\sum_{k=0}^{n}C_n^k\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-k+1)}\frac{\Gamma(a+n-\sigma+1)}{\Gamma(a-\sigma+k+1)},$$

and consequently

$$\sum_{k=0}^{n}\frac{1}{k!(n-k)!\Gamma(\sigma-k+1)\Gamma(a-\sigma+k+1)} = \frac{\Gamma(a+n+1)}{n!\Gamma(\sigma+1)\Gamma(a+1)\Gamma(a+n-\sigma+1)}.$$

This implies

$$\lambda_n = \sum_{k=0}^{n}\frac{\alpha_k}{(n-k)!}$$

$$= \Gamma(a+1)\Gamma(\sigma+1)\sum_{k=0}^{n}\frac{1}{k!(n-k)!\Gamma(\sigma-k+1)\Gamma(a-\sigma+k+1)}$$

$$= \frac{1}{n!}\frac{\Gamma(a+n+1)}{\Gamma(a+n-\sigma+1)}$$

and hence

$$e^{-t}\varrho(t) = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\frac{\Gamma(a+n+1)}{\Gamma(a+n-\sigma+1)}.$$

We therefore have

$$J_\sigma^* t^{a-\sigma}e^{-t}\varrho(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(a+n+1)}{\Gamma(a+n-\sigma+1)}J_\sigma^* t^{a-\sigma+n}.$$

But

$$J_\sigma^* t^{a-\sigma+n} = \frac{1}{\Gamma(\sigma)}\int_0^t(t-u)^{\sigma-1}u^{a-\sigma+n}du = t^{a+n}\frac{\Gamma(a-\sigma+n+1)}{\Gamma(a+n+1)}.$$

Finally we get

$$J_\sigma^* t^{a-\sigma}e^{-t}\varrho(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{a+n} = t^a e^{-t},$$

which implies the validity of (2.5) and (2.7).

Now take for ζ in (2.1) the function for which

$$\zeta'(t) = f(t) = \frac{t^{-s-1}}{\Gamma(\gamma)\Gamma(\sigma-s)}\int_0^t\eta^{\gamma-1}e^{-\eta}\varrho(\eta)\,d\eta,$$

where $a = \sigma-s+\gamma-1$, and the parameter γ satisfies the condition $\gamma > s$. We have

$$\bigvee \zeta = \|f\|_L \leqslant \frac{1}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^\infty t^{-s-1} \int_0^t \eta^{\gamma-1} e^{-\eta} |\varrho(\eta)|\, d\eta\, dt \tag{2.8}$$

$$= \frac{1}{s\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^\infty \eta^{\gamma-s-1} e^{-\eta} |\varrho(\eta)|\, d\eta$$

$$\leqslant \frac{1}{s\Gamma(\gamma)\Gamma(\sigma-s)} \sum_{k=0}^\infty |\alpha_k| \int_0^\infty \eta^{\gamma-s+k-1} e^{-\eta} d\eta$$

$$= \frac{\Gamma(a+1)\Gamma(\sigma+1)}{s\Gamma(\gamma)\Gamma(\sigma-s)} \sum_{k=0}^\infty \frac{1}{k!\,|\Gamma(\sigma-k+1)|}.$$

Evaluate the sum of the latter series. Show that

$$R = \sum_{k=0}^\infty \frac{1}{k!\,|\Gamma(\sigma-k+1)|} < \frac{2^{\sigma+1}}{\Gamma(\sigma+1)}. \tag{2.9}$$

If σ is a positive integer, then

$$R = \frac{2^\sigma}{\Gamma(\sigma+1)} \tag{2.10}$$

and therefore only the case of non-integer σ should be considered.

We have

$$(1+\tau)^\sigma = \sum_{k=0}^\infty \frac{\Gamma(\sigma+1)}{k!\,\Gamma(\sigma-k+1)} \tau^k, \quad |\tau| \leqslant 1.$$

Hence

$$\sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\sigma-k+1)} = 0, \quad \sum_{k=0}^\infty \frac{1}{k!\,\Gamma(\sigma-k+1)} = \frac{2^\sigma}{\Gamma(\sigma+1)}. \tag{2.11}$$

Furthermore, as $\Gamma(\sigma-k+1) > 0$ for $k \leqslant \sigma$ and $(-1)^{[\sigma]-k+1}\Gamma(\sigma-k+1) > 0$ for $k > \sigma$, we get

$$R = \sum_{k\leqslant\sigma} \frac{1+(-1)^{[\sigma]-k}}{\Gamma(\sigma-k+1)k!} + \sum_{k=0}^\infty \frac{(-1)^{[\sigma]+k+1}}{k!\,\Gamma(\sigma-k+1)}.$$

By (2.11) the last sum on the right equals 0 and therefore

$$R \leqslant 2\sum_{k\leqslant\sigma} \frac{1}{k!\,\Gamma(\sigma-k+1)}.$$

The sum of the series

$$r = \sum_{k>\sigma} \frac{1}{k!\,\Gamma(\sigma-k+1)}$$

is positive, since all the terms $\beta(k)$ of this series have alternative signs, the first term $\beta([\sigma]+1) > 0$ and the ratio

$$\nu_k = \left|\frac{\beta(k+1)}{\beta(k)}\right| = \frac{k-\sigma}{k+1} < 1.$$

Hence

$$R < 2\sum_{k=0}^\infty \frac{1}{k!\,\Gamma(\sigma-k+1)} = \frac{2^{\sigma+1}}{\Gamma(\sigma+1)}.$$

From inequalities (2.8), (2.9) we finally obtain

$$\bigvee \zeta = \|f\|_L \leqslant \frac{2^{\sigma+1}\Gamma(\sigma+1)}{s\Gamma(\gamma)\Gamma(\sigma-s)}, \quad \gamma > s,\ a = \sigma+\gamma-s-1. \tag{2.12}$$

Now find the function $(J_\sigma^*\zeta)' = J_\sigma^* f$. We have

$$f(t) = \frac{t^{\gamma-s-1}}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^1 \eta^{\gamma-1} e^{-\eta t} \varrho(\eta t)\, d\eta$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^1 \eta^s (t\eta)^{\gamma-s-1} e^{-\eta t} \varrho(\eta t)\, d\eta.$$

Hence, by (2.5)

$$(J_\sigma^* f)(t) = \frac{1}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^1 \eta^{s-\sigma} (t\eta)^{\sigma+\gamma-s-1} e^{-\eta t} d\eta$$

$$= \frac{t^{\sigma-s-1}}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^t \eta^{\gamma-1} e^{-\eta} d\eta = \theta(t) \frac{1}{\Gamma(\gamma)} \int_0^t \eta^{\gamma-1} e^{-\eta} d\eta.$$

Thus

$$(\theta - J_\sigma^* f)(t) = \frac{t^{\sigma-s-1}}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_t^\infty \eta^{\gamma-1} e^{-\eta} d\eta$$

and consequently

$$\|\theta - J_\sigma^* f\|_L = \frac{1}{\Gamma(\gamma)\Gamma(\sigma-s)} \int_0^\infty t^{\sigma-s-1} \int_t^\infty \eta^{\gamma-1} d\eta\, dt \tag{2.13}$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\sigma-s+1)} \int_0^\infty \eta^{\sigma+\gamma-s-1} e^{-\eta} d\eta = \frac{\Gamma(\sigma+\gamma-s)}{\Gamma(\gamma)\Gamma(\sigma-s+1)}.$$

Relations (2.12), (2.13) imply the following inequality for (2.1):

$$\varepsilon\left(\frac{2^{\sigma+1}\Gamma(\sigma+\gamma-s)}{s\Gamma(\gamma)\Gamma(\sigma-s)}\right) < \frac{\Gamma(\sigma+\gamma-s)}{\Gamma(\gamma)\Gamma(\sigma-s+1)}, \quad \gamma > s. \tag{2.14}$$

By (2.3) the function $\varepsilon(N)$ is a continuous function of N. Therefore, passing in (2.14) to the limit when $\gamma \to s$, we get (2.4). Lemma 1 is proved.

Let us now obtain an estimate of $\varepsilon(N)$ from below.

LEMMA 2. *For an arbitrary* $h > 0$,

$$\varepsilon\left(2h^{-s}\frac{\Gamma(\sigma)}{\Gamma(\sigma-s)}\right) \geqslant \frac{s}{\sigma}\,\frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)}. \tag{2.15}$$

Proof. If $\bigvee \zeta < \infty$, then the limit $\lim\limits_{t\to\infty} \zeta(t) = \zeta(\infty)$ exists and is finite. And if $c = \zeta(\infty) \neq 0$, then for $t \to \infty$

$$(J_\sigma^*\zeta)(t) \approx \frac{c}{\Gamma(\sigma+1)}t^\sigma.$$

But in this case

$$\|\theta-(J_\sigma^*\zeta)'\| \geqslant \lim_{t\to\infty}\left\{|(J_\sigma^*\zeta)(t)| - \frac{t^{\sigma-1}}{\Gamma(\sigma-s)}\right\} = \infty.$$

Therefore in (2.1) the considerations may be restricted to the functions ζ with $\zeta(\infty) = 0$. But if $\zeta(0) = \zeta(\infty) = 0$ and $\bigvee \zeta \leqslant N$, then $\|\zeta\|_{L_\infty} \leqslant N/2$. For such a function we have

$$|(J_\sigma^*\zeta)(t)| \leqslant \frac{N}{2\Gamma(\sigma+1)}t^\sigma, \quad t \geqslant 0.$$

Let the parameter $h > 0$ be connected with N by the relation $N = 2h^{-s}\dfrac{\Gamma(\sigma)}{\Gamma(\sigma-s)}$, and suppose that a function ζ satisfies the conditions $\zeta(0) = \zeta(\infty) = 0$, $\bigvee \zeta \leqslant N$. Then we have

$$\|\theta-(J_\sigma^*\zeta)'\|_{L(0,\infty)} \geqslant \int_0^h \{\theta(t)-(J_\sigma^*\zeta)'(t)\}\,dt = \frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)} - (J_\sigma^*\zeta)(h)$$

$$\geqslant \frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)} - \frac{N}{2\Gamma(\sigma+1)}h^\sigma = \frac{s}{\sigma}\,\frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)},$$

which proves the lemma.

Let us show that for $s \leqslant 1$, $\sigma \leqslant 2$ the inequality (2.15) turns into an equality. This will be done by means of a concrete function ζ which is constructed below.

For an arbitrary $h > 0$ define

$$\varrho_h(t) = \frac{t^{\sigma-1}}{\Gamma(\sigma-s)}\max\{0, t^{-s}-h^{-s}\}. \tag{2.16}$$

Consider the equation

$$\theta-(J_\sigma^*\zeta_h)' = \varrho_h, \quad \theta(t) = \frac{t^{\sigma-s-1}}{\Gamma(\sigma-s)} \tag{2.17}$$

with respect to the function ζ_h on the half-line $(0, \infty)$. It is not difficult to verify directly that (for $0 < s \leqslant 1$, $s < \sigma \leqslant 2$) the solution of (2.17) is the function ζ_h with the following properties. On the interval $(0, h]$,

$$\zeta_h(t) = \frac{h^{-s}}{\Gamma(\sigma-s)}. \tag{2.18}$$

On the half-line (h, ∞)

$$\zeta_h(t) = \frac{1}{\Gamma(\sigma-s)\Gamma(1-\sigma)}\left\{h^{-s}\int_0^{h/t}(1-u)^{-\sigma}u^{\sigma-1}du + t^{-s}\int_{h/t}^1 (1-u)^{-\sigma}u^{\sigma-s-1}du\right\} \tag{2.19}$$

if $0 < \sigma < 1$; $\zeta_h(t) = \theta(t)$ if $\sigma = 1$;

$$\zeta_h(t) = \frac{1}{\Gamma(\sigma-s)\Gamma(1-\sigma)}\left\{h^{-s}\int_0^{h/t}(1-u)^{1-\sigma}u^{\sigma-1}du + (1-s)t^{-s}\int_{h/t}^1 (1-u)^{1-\sigma}u^{\sigma-s-1}du\right\}$$

if $0 < s \leqslant 1 < \sigma < 2$, and finally,

$$\zeta_h(t) = \frac{1-s}{\Gamma(2-s)}t^{-s}$$

if $0 < s \leqslant 1$, $\sigma = 2$. By means of differentiation of these expressions for ζ_h we see that $\zeta_h'(t) < 0$ for $t \in (h, \infty)$. In addition,

$$\zeta_h(h+0) \leqslant \zeta_h(h-0) = \frac{h^{-s}}{\Gamma(\sigma-s)}.$$

Therefore ζ_h is non-increasing and non-negative on $(0, \infty)$ and $\zeta_h(\infty) = 0$.

In all cases we let $\zeta_h(0) = 0$. Then we get

$$\bigvee_0^\infty \zeta_h = 2h^{-s}\frac{\Gamma(\sigma)}{\Gamma(\sigma-s)}, \tag{2.20}$$

$$\|\theta-(J_\sigma^*\zeta_h)'\|_L = \|\varrho_h\|_L = \frac{s}{\sigma}\,\frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)}.$$

These relations, definition (2.1) of the quantity $\varepsilon(N)$, and inequality (2.15) imply the validity of the following assertion:

LEMMA 3. *If* $s \leqslant 1$, $s < \sigma \leqslant 2$, *then for any* $h > 0$

$$\varepsilon(N_h) = \frac{s}{\sigma}\,\frac{h^{\sigma-s}}{\Gamma(\sigma-s+1)}, \quad N_h = 2h^{-s}\frac{\Gamma(\sigma)}{\Gamma(\sigma-s)}. \tag{2.21}$$

3

Consider now some properties of the operator D_σ and of the class W_σ defined in Section 1.

Since $\varepsilon(N) < \infty$, there exists a function ζ with the properties as follows:

$$\bigvee \zeta < \infty, \quad \zeta(0) = 0, \quad \|\theta - (J_\sigma^* \zeta)'\|_L < \infty. \tag{3.1}$$

Using the function ζ, define the operator F on W_σ by the relation

$$(Fx)(t) = \int_0^\infty x(t+\eta)\, d\zeta(\eta) + \int_0^\infty (D_\sigma x)(t+\eta)\varrho(\eta)\, d\eta, \tag{3.2}$$

where $\varrho = \theta - J_\sigma^* \zeta$. It is clear that $Fx \in C$.

Lemma 4. *For an arbitrary function $x \in W_\sigma$ and ζ with the properties* (3.1) *the following identity holds*:

$$Fx = D_s x. \tag{3.3}$$

The following assertion should be proved:

$$\int_0^\infty x D_\sigma^* \varphi\, dt = \int_0^\infty \varphi Fx\, dt \quad \forall \varphi \in K. \tag{3.4}$$

It follows from definition (1.6) of the function $D_\sigma x$ that for any $t \in [0, \infty)$

$$\int_0^\infty (D_\sigma x)(\xi + t)\varphi(\xi)\, d\xi = \int_0^\infty x(\xi + t)(D_\sigma^* \varphi)(\xi)\, d\xi.$$

Hence

$$\int_0^\infty \varphi Fx\, dt = \int_0^\infty x(\xi) \int_0^\xi \varphi(\xi - \tau)\, d\zeta(\tau)\, d\xi + \int_0^\infty x(\xi) \int_0^\xi \varrho(\tau)(D_\sigma^* \varphi)(\xi - \tau)\, d\tau\, d\xi.$$

Introducing this expression into (3.4), we see that the following relation should be checked:

$$(D_\sigma^* \varphi)(\xi) - \int_0^\xi \varphi(\xi - \tau)\, d\zeta(\tau) - \int_0^\xi \varrho(\tau)(D_\sigma^* \varphi)(\xi - \tau)\, d\tau = 0 \tag{3.5}$$

for $\xi \in (0, \infty)$. In the second integral substitute $\varrho = \theta - (J_\sigma^* \zeta)'$; this implies

$$\int_0^\xi \varrho(\tau)(D_\sigma^* \varphi)(\xi - \tau)\, d\tau = (J_{\sigma-s}^* D_\sigma^* \varphi)(\xi) - \int_0^\xi (J_\sigma^* \zeta)'(\tau)(D_\sigma^* \varphi)(\xi - \tau)\, d\tau. \tag{3.6}$$

We have $J_{\sigma-s}^* D_\sigma^* \varphi = D_s^* \varphi$. The second integral (let us denote it by A) in (3.6) may be represented in the form

$$A = \int_0^\xi \varphi(\xi - \tau)\, d\zeta(\tau). \tag{3.7}$$

But (3.6)–(3.7) give (3.5), which proves the lemma.

Theorem 1. *In the case $0 < s < \sigma$ the relations*

$$W_\sigma \subset W_s, \quad D_s W_\sigma \subset C \tag{3.8}$$

hold, and for arbitrary $N > 0$, $\delta > 0$ the estimate

$$\omega(\delta) \leqslant \varepsilon(N) + N\delta < \infty \tag{3.9}$$

is valid.

Proof. Embeddings (3.8) follow from representations (3.2), (3.3) and Lemma 1. Furthermore (3.2)–(3.3) give the inequality

$$|(D_s x)(t)| \leqslant \|x\|_C \bigvee \zeta + \|D_\sigma x\|_{L_\infty} \|\theta - (J_\sigma \zeta)'\|_L,$$

which implies (3.9).

From definition (1.7) of the quantity $\omega(\delta)$, from the its finiteness, and formula (1.11) it follows that for functions from the class W_σ the inequality

$$\|D_s x\|_C \leqslant G \|x\|_C^{1-s/\sigma} \|D_\sigma x\|_{L_\infty}^{s/\sigma}, \quad G = \omega(1), \tag{3.10}$$

holds.

Relations (1.11), (3.9) give the following estimate for G:

$$G \leqslant \varepsilon(N)\, \delta^{-(\sigma-s)/\sigma} + N\delta^{s/\sigma}.$$

Minimizing the right-hand side of this relation in $\delta > 0$, we obtain

$$G = \omega(1) \leqslant \left(\frac{s}{\sigma}\right)^{-s/\sigma} \left(\frac{\sigma - s}{\sigma}\right)^{s/\sigma - 1} N^{1-s/\sigma} \{\varepsilon(N)\}^{s/\sigma}. \tag{3.11}$$

Statement (2.4) of Lemma 1 and inequality (3.11) give the following estimate from above for G:

$$G \leqslant \frac{\Gamma(\sigma+1)}{\Gamma(s+1)\Gamma(\sigma-s+1)} 2^{\frac{\sigma+1}{\sigma}(\sigma-s)}. \tag{3.12}$$

An estimate from below of the constant G may be obtained by means of any function $x \in W_\sigma$. We use the function

$$u(t) = \begin{cases} 2(1-t)^\sigma - 1, & t \in [0, 1], \\ -1, & t \in (1, \infty) \end{cases} \tag{3.13}$$

for this purpose.

Lemma 5. *If u is defined by* (3.13), *then $u \in W_\sigma$ and*

$$\|u\|_C = 1, \quad \|D_\sigma x\|_{L_\infty} = 2\Gamma(\sigma+1), \qquad \|D_s x\|_C = (D_s x)(0) = \frac{2\Gamma(\sigma+1)}{\Gamma(\sigma-s+1)}. \tag{3.14}$$

Proof. We first verify that the function $X = c = \text{const}$ belongs to W_σ and $D_\sigma c \equiv 0$. It is so for integer values of σ. Let σ be non-integer. By virtue of (1.6), we should prove that

$$\int_0^\infty D_\sigma^* \varphi\, dt = 0 \quad \forall \varphi \in K. \tag{3.15}$$

For any positive integer $m > \sigma$ we have $D_\sigma^* \varphi = J_{m-\sigma}^* \varphi^{(m)} = (J_{m-\sigma}^* \varphi^{(m-1)})'$. The function $\psi = J_{m-\sigma}^* \varphi^{(m-1)}$ possesses the properties $\psi(0) = 0$ and $\psi(t) = O(t^{-\sigma})$ for $t \to \infty$. Thus

$$\int_0^\infty D_\sigma^* \varphi\, dt = \int_0^\infty \psi'\, dt = 0,$$

i.e. (3.15) holds.

Consider now the function

$$x_\alpha(t) = \frac{1}{\Gamma(\alpha+1)}(1-t)_+^\alpha,$$

where $(1-t)_+ = \max\{1-t, 0\}$, $\alpha \geqslant 0$; it is clear that $x_\alpha \in M$. Further we have $J_\beta x_\alpha = x_{\alpha+\beta}$ for any $\alpha \geqslant 0$, $\beta > 0$ and thus $D_\beta x_{\alpha+\beta} = x_\alpha$. Hence $x_\sigma \in W_\sigma$, $D_\sigma x_\sigma = x_0$ and $D_s x_\sigma = x_{\sigma-s}$.

The function (3.13) may be written in the form $u = 2\Gamma(\sigma+1)x_0 - 1$ and thus $u \in W_\sigma$. Properties (3.14) are verified quite simply and our lemma follows.

The function u gives the following estimate from below for G:

$$G \geqslant \frac{D_s u(0)}{\|u\|_C^{1-s/\sigma}\|D_\sigma u\|_{L_\infty}^{s/\sigma}} = \frac{\{2\Gamma(\sigma+1)\}^{1-s/\sigma}}{\Gamma(\sigma-s+1)}. \tag{3.16}$$

Let us combine inequalities (3.12), (3.16) in one statement.

THEOREM 2. *For the best constant G in* (3.10) *the following estimates hold*:

$$\frac{\{2\Gamma(\sigma+1)\}^{1-s/\sigma}}{\Gamma(\sigma-s+1)} \leqslant G \leqslant \frac{\Gamma(\sigma+1)}{\Gamma(s+1)\Gamma(\sigma-s+1)} 2^{\frac{\sigma+1}{\sigma}}. \tag{3.17}$$

Estimates of the constant G for positive integer values of the parameters s, σ were treated in papers by A. Gorny, H. Cartan, A. P. Matorin, V. M. Olovjaniohnikov and others. For detailed references see Stechkin [14].

A function $\bar{x} \in W_\sigma$ for which (3.10) turns into an equality is called *extremal.* It is plain that together with $\bar{x}$ any function of the form $c\bar{x}(ht)$ is also extremal.

THEOREM 3. *If* $0 < s \leqslant 1$, $s < \sigma \leqslant 2$, *then the best constant in* (3.10) *is*

$$G = \frac{\{2\Gamma(\sigma+1)\}^{1-s/\sigma}}{\Gamma(\sigma-s+1)}, \tag{3.18}$$

and the function u defined by (3.13) *is extremal.*

In the case $s = 1$, $\sigma = 2$ we have $G = 2$, and (3.10) coincides with the Landau–Hadamard inequality (1.2)

Proof. Statements (2.12) of Lemma 3 and inequality (3.11) imply the estimate

$$G \leqslant \frac{\{2\Gamma(\sigma+1)\}^{1-s/\sigma}}{\Gamma(\sigma-s+1)}. \tag{3.19}$$

The statements of the theorem follow from inequalities (3.16), (3.19).

4

There are some values of parameters s, σ for which formulas (2.21), (3.18) are not valid. To prove this we compute the quantities $\varepsilon(N)$, G, in the case $\sigma = 2$, $1 \leqslant s < 2$.

As above, let $\theta(t) = \frac{1}{\Gamma(2-s)} t^{1-s}$, $t \in (0, \infty)$; $h > 0$. Put $b = 1+\sqrt{2}$ and $a = \frac{1+\sqrt{2}}{\sqrt{2}}$. Define the function ζ_h on the half-line $[0, \infty)$ by the relations

$$\begin{aligned} &\zeta_h(0) = 0, \\ &\zeta_h(t) = \frac{1}{h}\{2\theta(ah) - \theta(bh)\}, && t \in (0, h], \\ &\zeta_h(t) = \frac{\sqrt{2}}{h}\{\theta(bh) - \theta(ah)\}, && t \in (h, bh], \\ &\zeta_h(t) = \theta'(t), && t \in (bh, \infty). \end{aligned} \tag{4.1}$$

The function θ is non-increasing on $(0, \infty)$; therefore $\zeta_h > 0$ on $(0, h]$ and $\zeta_h < 0$ on $(h, bh]$. Further, since

$$\frac{\sqrt{2}}{h}\{\theta(bh) - \theta(ah)\} = \theta'(\xi), \qquad \xi \in (ah, bh)$$

and since θ' is non-increasing, the function ζ is non-decreasing on (h, ∞). Moreover, we have $\zeta_h(\infty) = 0$. Hence

$$N_h = \bigvee_0^\infty \zeta_h = 2\zeta_h(+0) - 2\zeta_h(h+0) = \frac{2h^{-s}}{\Gamma(2-s)}(1+\sqrt{2})^{2-s}(2^{s/2}-1). \tag{4.2}$$

Consider now the function $\lambda_h = J_1^*\zeta_h = (J_2^*\zeta_h)'$. It is not difficult to verify that λ_h is continuous on $[0, \infty)$, λ_h linear on the intervals $[0, h]$, $[h, bh]$; $\lambda_h(0) = 0$, $\lambda_h(ah) = \theta(ah)$; λ_h coincides with θ on $[bh, \infty)$, $\lambda(t) \leqslant \theta(t)$ for $t \in [0, ah]$, $\lambda(t) \geqslant \theta(t)$ for $t \in [ah, bh]$. Furthermore, elementary calculations give the following value for the quantities $\|\theta - \lambda_h\|_L$:

$$\varepsilon_h = \|\theta - \lambda_h\|_L = \|\theta - (J_2^*\zeta_h)'\|_L = \frac{sh^{2-s}}{2\Gamma(3-s)}(1+\sqrt{2})^{2-s}(2^{s/2}-1). \tag{4.3}$$

Hence the inequality

$$\varepsilon(N_h) \leqslant \varepsilon_h \tag{4.4}$$

follows.

We shall now obtain the estimate from below for $G = \omega(1)$ and hence (by (3.11)) also for $\varepsilon(N)$. For the latter purpose consider the function

$$u(t) = \begin{cases} 2(t-1)^2 - 1, & t \in [0, a], \\ 1 - 2(t-b)^2, & t \in [a, b], \\ 1, & t \geqslant b. \end{cases} \tag{4.5}$$

We have $u \in W_2$, $\|u\|_C = 1$, $\|u''\|_{L_\infty} = 4$. The functions u'' has a compact support and thus $u'' \in M$. Therefore $u = 1 + 4J_2 u''$ and $D_s u = J_{2-s}u''$. Hence

$$(D_s u)(0) = \frac{1}{\Gamma(2-s)} \int_0^\infty t^{1-s} u''(t) = \frac{4}{\Gamma(3-s)}(1+\sqrt{2})^{2-s}(2^{s/2}-1).$$

By virtue of (3.10) this implies the following estimate for G:

$$G = \omega(1) \geqslant \frac{2^{2-s}}{\Gamma(3-s)}(1+\sqrt{2})^{2-s}(2^{s/2}-1). \tag{4.6}$$

Using (3.11), we see that actually equality holds in (4.4) and (4.6). Thus we have the following statements:

If $1 \leqslant s < 2$, $\sigma = 2$, *then*

$$\varepsilon(N_h) = \varepsilon_h, \tag{4.7}$$

where the quantities ε_h *and* N_h *are defined by formulas* (4.3) *and* (4.2).

If $1 \leqslant s < 2$, *then the inequality*

$$\|D_s x\|_C \leqslant \frac{2^{2-s}}{\Gamma(3-s)}(1+\sqrt{2})^{2-s}(2^{\frac{s}{2}}-1)\|x\|_C^{\frac{2-s}{2}}\|x''\|_{L_\infty}^{\frac{s}{2}}$$

holds for functions from the class W_2 *and the function* (4.5) *is extremal.*

For $s = 1$ we again obtain the Landau–Hadamard inequality (1.2) but with another extremal function.

5

Consider now problem (1.8). Let

$$U(T) = \sup_{x\in Q}\|D_s x - Tx\|_C, \quad E(N) = \inf_{\|T\|_C^C \leqslant N} U(T), \tag{5.1}$$

where $Q = \{x \in W_\sigma: \|D_\sigma x\|_{L_\infty} \leqslant 1\}$. We now show that problems (5.1) and (2.1) are equivalent.

THEOREM 4. $E(N) = \varepsilon(N)$.

Proof. Let the function ζ satisfy the conditions $\bigvee \zeta \leqslant N$, $\zeta(0) = 0$ and $\varrho = \theta - (J_\sigma\zeta)' \in L$. Define the operator T_ζ by the following relation:

$$(T_\zeta x)(t) = (\zeta * x)(t) = \int_0^\infty x(t+\eta)\,d\zeta(\eta). \tag{5.2}$$

It is obvious that $\|T_\zeta\|_C^C = \bigvee \zeta \leqslant N$. Furthermore, by virtue of Lemma 4 we have

$$(D_s x - T_\zeta x)(t) = \int_0^\infty \varrho(\eta)\,(D_\sigma x)\,(t+\eta)\,d\eta.$$

Therefore, $E(N) \leqslant U(T_\zeta) \leqslant \|\varrho\|_L$, and hence the inequality

$$E(N) \leqslant \varepsilon(N) \tag{5.3}$$

follows.

Using (5.3) and (1.12), we have

$$\Omega(N) \leqslant E(N) \leqslant \varepsilon(N).$$

In the case $0 < s \leqslant 1$, $s < \sigma \leqslant 2$, by means of Theorem 3 and Lemma 3 we verify that $\Omega(N) = \varepsilon(N)$ and consequently $E(N) = \varepsilon(N)$.

We now prove that the inequality $E(N) \geqslant \varepsilon(N)$ holds for $\sigma > 1$. Denote by C_0 the set of continuous functions with compact supports on $[0, \infty)$. It is obvious

that $R = J_\sigma C_0 \subset W_\sigma \cap C_0$. Put $Q_0 = \{x \in J_\sigma C_0: \|D_\sigma x\|_C \leqslant 1\}$. Let the linear operator T be such that $\|T\|_C^C \leqslant N$ and $U(T) < \infty$. Put

$$u(T) = \sup_{x\in Q_0}(D_s x - Tx)(0).$$

It is plain that $u(T) \leqslant U(T) < \infty$. On the set C_0 we have

$$(Tx)(0) = \int_0^\infty x\,d\zeta, \tag{5.4}$$

where $\bigvee \zeta \leqslant \|T\|_C^C \leqslant N$ and we may suppose that $\zeta(0) = 0$. Let $x \in J_\sigma C_0$ and hence $x = J_\sigma y$, where $y \in C_0$. Then

$$(Tx)(0) = \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_t^\infty (\tau - t)^{\sigma-1}y(\tau)\,d\tau\,d\zeta(t)$$
$$= \frac{1}{\Gamma(\sigma)}\int_0^\infty y(\tau)\int_0^\tau (\tau-t)^{\sigma-1}d\zeta(t)\,d\tau = \int_0^\infty yJ_{\sigma-1}^*\zeta\,d\tau.$$

Besides, we have

$$(D_s x)(0) = (J_{\sigma-s}^* D_\sigma x)(0) = \int_0^\infty \theta y\,dt.$$

Consequently,

$$(D_s x - Tx)(0) = \int_0^\infty (\theta - J_{\sigma-1}^*\zeta)D_\sigma x\,dt, \quad x \in J_\sigma C_0.$$

And since $y = D_\sigma x$ is an arbitrary function from the class C_0, we have

$$u(T) = \|\theta - J_{\sigma-1}^*\zeta\|_{L(0,\infty)} \leqslant U(T).$$

Hence $\varepsilon(N) \leqslant U(T)$ and $\varepsilon(N) \leqslant E(N)$, which completes the proof of our theorem.

It may be seen from the proof of the theorem that, if the lower bound in (2.1) is attained for a function ζ, then an extremal operator of problem (5.1) corresponds to the function defined by (5.2). Conversely, an extremal function of the problem (2.1) corresponds by (5.4) to each extremal operator T of the problem (5.1).

Along with $E(N)$ define the quantity

$$e(N) = \inf_{\|T\|_{C^*} \leqslant N}\sup_{x\in Q}\{(D_s x)(0) - Tx\}. \tag{5.5}$$

Stechkin's inequality holds for $e(N)$, just as for $E(N)$:

$$e(N) \geqslant \Omega(N). \tag{5.6}$$

Repeating the arguments used in Theorem 4 we verify that $\varepsilon(N) = e(N)$. It follows from the results of Gabushin [16] that $e(N) = \Omega(N)$, i.e. (5.6) turns into an equality. Thus

$$E(N) = e(N) = \varepsilon(N) = \frac{s}{\sigma}\left(\frac{\sigma - s}{\sigma}\right)^{(\sigma-s)/\sigma} G^{s/\sigma}N^{-(\sigma-s)/s}, \tag{5.7}$$

where $G = \omega(1)$; in particular, the calculation of the best constant G in inequality (3.10) may always be reduced to the problem of the best approximation in $L(0,\infty)$ of the function θ by functions $J^*_{\sigma-1}\zeta$.

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ON THE UNIFORM CONTINUITY OF METRIC PROJECTION

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Let X denote a normed space, M a convex set in X, for $x \in X$, $xM = \inf\{||x-y||: y \in M\}$ the distance from x to M, $x_M = \{y \in M: ||x-y|| = xM\}$ the set of the elements from M of the best approximation for x. In what follows, M is assumed to be an existence set [9], i.e. x_M is nonempty for any $x \in X$. A map $x \to x_M$ (set-valued, in general) is called a *metric projection*. It is well known that the metric projection is single-valued for any existence set in the strictly convex space.

In Section 1 there is considered a problem of estimating $||x_M - y_M||$ from above via $||x-y||$ and geometric characteristics of X under the condition of strict convexity. Section 2 is devoted to the metric projection from $L(S,\mu)$ onto its finite-dimensional subspace. The main results are contained in Theorems 3 and 4.

1

Further we will denote by θ the zero of the space X and $V(x,r) = \{y \in X: ||x-y|| \leqslant r\}$ $(r > 0)$, $V = V(\theta, 1)$. Let $d(M) = \sup\{||x-y||: x, y \in M\}$ be the diameter of the set $M \subset X$. For a given set M contained in the plane $P \subset X$, we denote by $s(M)$ the width of M with respect to P, namely

$$s(M) = s(M)_P = \inf\{\sup_{x\in M-p} f(x) - \inf_{x \in M-p} f(x): f \in (P-p)^*,\ ||f|| = 1\},$$

where p is an element from P and $(P-p)^*$ is a conjugate space to $(P-p)$. Assume [1]

$$(1) \qquad \Omega(t) = \Omega(t)_X = \sup d(M \cap V) \quad (t \geqslant 0),$$

where the supremum is taken over all the hyperplanes $P \subset X$ such that $s(P \cap V)_P \leqslant t$. The function $\Omega(t)$ is nondecreasing and continuous in $(0, 2]$.

Let us estimate $||x_M - y_M||$ by means of Ω in the case of a strictly convex X.

LEMMA 1. *Let $M \subset X$ be the convex existence set, $x^1, x^2 \notin M$, $||x^1_M - x^2_M|| > 0$; then the inequalities*

$$(2) \qquad ||x^1_M - x^2_M|| \leqslant ||x^1_L - x^2_L||, \quad x^iM - 2||x^1-x^2|| \leqslant x^iL \leqslant x^iM \quad (i = 1, 2)$$

hold for the line $L = \{(1-\lambda)x^1_M + \lambda x^2_M: |\lambda| < \infty\}$.

Proof. The first inequality and the inequalities $x^iL \leqslant x^iM$ $(i = 1, 2)$ were established in [1] (see also [10]). It is easily seen that the element $z \in [x^1, x^2]$ with